ELEMENTARY ABELIAN 2-GROUPS THAT ACT FREELY ON PRODUCTS OF REAL PROJECTIVE SPACES

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ABSTRACT. For a natural number N let $\bar{N}$ be 0 if N is even and 1 if N is odd. We prove that if $(\mathbf{Z}_2)^l$ acts freely on $\prod_{j=1}^k \mathbf{R}P^{N_j}$ in such a way that the induced action on mod 2 cohomology is trivial, then $l \leq 2(\bar{N}_1 + \cdots + \bar{N}_k)$. If no N_j is congruent to 3 mod 4 then $l \leq \bar{N}_1 + \cdots + \bar{N}_k$.

In this note we adapt some techniques developed by G. Carlsson [1] to prove certain restrictions on elementary abelian 2-groups that may act freely on products of real projective spaces.

For m a natural number, let

$$\bar{m} = \begin{cases} 0, & m \text{ even,} \\ 1, & m \text{ odd.} \end{cases}$$

MAIN THEOREM. *Suppose X is a finite CW-complex with the mod 2 cohomology of $\prod_{j=1}^k \mathbf{R}P^{N_j}$. Suppose further that X admits a free action by $\mathbf{Z}_2^l$ such that the induced action on mod 2 cohomology is trivial. Then (a) $l \leq 2(\bar{N}_1 + \cdots + \bar{N}_k)$, and (b) if no N_j is congruent to 3 mod 4 then $l \leq \bar{N}_1 + \cdots + \bar{N}_k$.*

We would like to remark on the sharpness of the above theorem. It is known that any $\mathbf{R}P^{2n+1}$ admits a free $\mathbf{Z}_2$ -action and $\mathbf{R}P^{4n+3}$ a free $\mathbf{Z}_2^2$ -action (the latter arises from the free quaternion action on S^{4n+3}). Thus, if each N_j is even or congruent to 3 mod 4, then $\prod_{j=1}^k \mathbf{R}P^{N_j}$ will admit a free $\mathbf{Z}_2^{2(\bar{N}_1 + \cdots + \bar{N}_k)}$ -action satisfying the above hypothesis. On the other hand, if each N_j is even or congruent to 1 mod 4, then $\prod_{j=1}^k \mathbf{R}P^{N_j}$ will admit a free $\mathbf{Z}_2^{\bar{N}_1 + \cdots + \bar{N}_k}$ -action.

In light of all this, there seems to be an obvious conjecture that is compatible with the above theorem. To state the conjecture we define a function μ on natural numbers by

$$\mu(m) = \begin{cases} 0 & \text{if } m \text{ even,} \\ 1 & \text{if } m \equiv 1 \pmod{4}, \\ 2 & \text{if } m \equiv 3 \pmod{4}. \end{cases}$$

CONJECTURE. *Suppose X and $\mathbf{Z}_2^l$ satisfy the hypothesis of the main theorem; then $l \leq \mu(N_1) + \cdots + \mu(N_k)$.*

We remark that $\prod_{j=1}^k \mathbf{R}P^{N_j}$ admits a free $\mathbf{Z}_2^{\mu(N_1) + \cdots + \mu(N_k)}$ -action.

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Suppose X is a finite CW-complex that admits an action by a group G . Let X_G denote $EG \times_G X$. Consider the mod 2 Serre spectral sequence associated to the fibration $X \rightarrow X_G \rightarrow BG$. It has E_2 -term

$$E_2^{**} \cong H^*(BG; H^*(X; \mathbb{Z}_2)).$$

PROPOSITION A. *If G acts freely on X then E_∞ is a finite-dimensional $\mathbb{Z}_2$ -vector space.*

PROOF. Under the hypothesis $EG \times_G X \simeq X/G$, which is a finite CW-complex. The proposition follows since E_∞^{**} is the associated graded groups to some filtration on $H^*(X/G; \mathbb{Z}_2)$ which is finite dimensional. $\square$

Now assume that X and $G = \mathbb{Z}_2^l$ satisfy the hypothesis of the Main Theorem, and consider the Serre spectral sequence associated to the fibration

$$X \rightarrow X_G \rightarrow BG.$$

Our E_2 term looks like

$$\begin{aligned} E_2^{**} &\cong H^*(BG; H^*(X; \mathbb{Z}_2)) \\ &\cong H^*(BG; \mathbb{Z}_2) \otimes H^*(X; \mathbb{Z}_2) \\ &\cong \mathbb{Z}_2[\gamma_1, \dots, \gamma_l] \otimes \mathbb{Z}_2[Z_1, \dots, Z_k] / (Z_1^{N_1+1}, \dots, Z_k^{N_k+1}), \end{aligned}$$

where $\dim \gamma_i = 1$ and $\dim Z_j = 1$.

Let $w_j = d_2(Z_j) \in H^2(BG; \mathbb{Z}_2)$, where d_2 is the first differential in the spectral sequence.

PROPOSITION B. (a) *If N_j is even, then $w_j = 0$, and (b) if $N_j \equiv 1 \pmod{4}$, then $\text{Sq}^1 w_j$ is an element of the ideal $(w_1, \dots, w_k)$ in $H^*(BG; \mathbb{Z}_2)$.*

PROOF. (a) Assume $N_j = 2m$. Then $0 = d_2(Z_j^{2m+1}) = Z_j^{2m} \otimes d_2(Z_j)$, since d_2 is a derivation. Now we just note that the map

$$Z_j^{N_j} \otimes \text{---} : E_2^{*,0} \rightarrow E_2^{*,N_j}$$

is an injection, so $d_2(Z_j) = 0$.

(b) Now assume $N_j = 4m + 1$. We again use the derivation property of the differentials to compute $0 = d_3(Z_j^{4m+2}) = Z_j^{4m} \cdot d_3(Z_j^2)$. Now $Z_j^2 = \text{Sq}^1 Z_j$ and the transgression operator commutes with the action of the Steenrod algebra, so $d_3(Z_j^2)$ is represented by $\text{Sq}^1 w_j$, and so, we must have that $Z_j^{4m} \otimes \text{Sq}^1 w_j \in E_2^{3,4m}$ is in the image of d_2 . But d_2 is completely determined by $d_2(Z_j) = w_j$ and the fact that it is a derivation. So we must have

$$Z_j^{4m} \otimes \text{Sq}^1 w_j = Z_j^{4m} \otimes d_2 f,$$

where $f \in E_2^{1,1}$. But $Z_j^{4m} \otimes \text{---} : E_2^{*,0} \rightarrow E_2^{*,4m}$ is an injection, so $\text{Sq}^1 w_j = d_2 f$. Now, write $f = \sum_{i,j} \lambda_{i,j} Z_i \otimes \gamma_j$, where each $\lambda_{i,j} \in \mathbb{Z}_2$. Then, we compute

$$d_2 f = \sum_{i,j} \lambda_{i,j} w_i \gamma_j,$$

which is clearly in the stated ideal. $\square$

PROPOSITION C [1]. Suppose $f_1, \dots, f_k$ are elements of $H^n(B\mathbf{Z}_2^l; \mathbf{Z}_2)$, regarded as homogeneous polynomials of degree n in l variables. Then they have a nontrivial common zero in $\mathbf{Z}_2^l$ if and only if there is a subgroup inclusion $i: \mathbf{Z}_2 \hookrightarrow \mathbf{Z}_2^l$ such that each $i^*f_j = 0$.

PROPOSITION D [3]. If $f_1, \dots, f_k$ are homogeneous polynomials in l variables of degree n with coefficients in $\mathbf{Z}_2$, and if $l > nk$, then they have a nontrivial common zero in $\mathbf{Z}_2^l$.

PROOF OF MAIN THEOREM. PART (a). We first claim that the degree 2 polynomials $w_1, \dots, w_k$ have no nontrivial common zero. If they did, then Proposition C would provide a subgroup inclusion $i: \mathbf{Z}_2 \hookrightarrow \mathbf{Z}_2^l$ for which each $i^*w_j = 0$. But if this were the case, we could restrict our G -action on X to this $\mathbf{Z}_2$, and the naturality of the Serre spectral sequence with respect to subgroup inclusions would imply that the sequence for the fibration $X \rightarrow X_{\mathbf{Z}_2} \rightarrow B\mathbf{Z}_2$ would collapse, $E_2 = E_\infty$, which is impossible since, by Proposition A, E_∞ must be finite dimensional. Since there can be no such restriction there can be no nontrivial common zero.

Part (a) of the Main Theorem now follows from Propositions D and B, part (a).

□

PROPOSITION E [1]. Let $(f_1, \dots, f_k)$ be an ideal generated by homogeneous polynomials f_j in $\mathbf{Z}_2[\gamma_1, \dots, \gamma_l]$ which is invariant under the action of the Steenrod algebra. Then if $l > k$, there exists a nontrivial common zero to $f_1, \dots, f_k$.

PROOF OF MAIN THEOREM. PART (b). The hypothesis and Proposition B imply that the ideal $(w_1, \dots, w_k)$ is invariant under the action of the Steenrod algebra. Proposition E implies that if $l > \bar{N}_1 + \dots + \bar{N}_k$, then there would exist a nontrivial common zero to $w_1, \dots, w_k$. But we have already seen, in the proof of part (a), that this is an impossibility. Thus, we must have $l \leq \bar{N}_1 + \dots + \bar{N}_k$. □

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