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ON WEAKLY $\mathcal{K}$-COUNTABLY DETERMINED SPACES OF CONTINUOUS FUNCTIONS

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ABSTRACT. A compact space K is said to be Gul'ko compact if the space $C(K)$ is $\mathcal{K}$-countably determined in the weak topology. Well-known compact sets, such as Eberlein compact sets, are Gul'ko compact. We prove here that the countable chain condition and metrizability are equivalent for Gul'ko compact sets.

The purpose of this note is to give a positive answer to a question of Talagrand (Problème 7.9 in **[13]**). In fact, we prove that: if K is a compact Hausdorff space, such that the Banach space $C(K)$ of all real-valued continuous functions on K is $\mathcal{K}$-countably determined in its weak topology, then in K there is a family of pairwise disjoint nonempty open sets of cardinality equal to the least cardinality of a base for K.

We denote by Σ the space of irrationals; Σ is identified with $\mathbf{N}^{\mathbf{N}}$. S denotes the set of finite sequences of natural numbers. For $s \in S$ we denote by $|s|$ the length (i.e., the domain) of s. For $s, t \in S$, we write $s \prec t$ if s is equal to the first $|s|$ terms of the sequence t. If $\sigma \in \Sigma$, and $n < \omega$, $\sigma | n$ denotes the finite sequence of the first n terms of σ. If $s \in S$, and $n < \omega$, $\widehat{s, n}$ denotes the finite sequence of length $|s| + 1$, whose first $|s|$ terms is s, and whose last term is n.

A space K is *Corson-compact* if K is homeomorphic to a subspace of

$$\Sigma\left([0,1]^{\Gamma}\right) = \left\{x \in [0,1]^{\Gamma} \colon \operatorname{supp}(x) \text{ is countable}\right\},$$

for some set Γ, where we have set $\operatorname{supp}(x) = \{\gamma \in \Gamma \colon x_\gamma \neq 0\}$ for $x = (x_\gamma)_{\gamma\in\Gamma} \in [0,1]^{\Gamma}$.

In a Corson-compact space K, the *weight* of K (i.e., the least cardinality of a base for K) is equal to the *density character* of K (i.e., to the least cardinality of a dense subset of K). Furthermore, every sequence in K has a convergent subsequence.

A topological space X satisfies the *countable chain condition* (c.c.c.) if every family of pairwise disjoint nonempty open subsets of X is countable (cf. **[5]**).

The following definition (in some equivalent form) has been introduced by Archangel'skii **[2]**, Vašák **[14]**.

DEFINITION. A topological space X is *$\mathcal{K}$-countably determined* if there are a subset Σ' of the space Σ of irrationals, a compact Hausdorff space K, and a closed subset F of $K \times \Sigma'$ such that (denoting by $\pi_1 \colon K \times \Sigma' \to K$ the projection $\pi_1(x, y) = x$) $X = \pi_1(F)$.

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If $\Sigma' = \Sigma$, then X is called *$\mathcal{K}$-analytic* (cf. [4]).

We need the following simple

LEMMA. *Let X be a $\mathcal{K}$-countably determined space. There is a family $\{A_s: s \in S\}$ of subsets of X such that $A_\varnothing = X$, $\bigcup_{k<\omega} A_{s^\frown k} = A_s$ for $s \in S$, and for every $x \in X$ there is a $\sigma \in \Sigma$ such that* (a) *$x \in \bigcap_{k<\omega} A_{\sigma|k}$ and* (b) *if $x_k \in A_{\sigma|k}$ for $k < \omega$ then the sequence (x_k) has a limit point in X.*

PROOF. There are $\Sigma' \subset \Sigma$, a compact space K, and a closed subset F of $K \times \Sigma'$ such that $\pi_1(F) = X$. There is a family $\{B_s: s \in S\}$ of subsets of Σ' such that $B_\varnothing = \Sigma'$, $\bigcup_{k<\omega} B_{s^\frown k} = B_s$ for $s \in S$, $\{B_s: |s| = n\}$ is a family of open-and-closed pairwise disjoint subsets of Σ', $\operatorname{diam}(B_s) < 1/|s|$ for $s \in S$, and

$$\begin{aligned}\bigcap_{k<\omega} B_{\sigma|k} &= \{\sigma\}, \qquad \sigma \in \Sigma',\\ &= \varnothing, \qquad \sigma \in \Sigma \setminus \Sigma'.\end{aligned}$$

We set $A_s = \pi_1(\pi_2^{-1}(B_s) \cap F)$ for $s \in S$. If $x \in X$, then there is $\sigma \in \Sigma'$ such that $(x, \sigma) \in F$. It is clear that $x \in \bigcap_{k<\omega} A_{\sigma|k}$. Let $x_k \in A_{\sigma|k}$ for $k < \omega$. Then there is $\sigma_K \in \Sigma'$ such that $(f_k, \sigma_k) \in \pi_2^{-1}(B_{\sigma|k}) \cap F$, hence $\sigma_k \in B_{\sigma|k}$ for $k < \omega$. Hence $\sigma_k \to \sigma$. Since K is compact and $X \subset K$ there is a limit point $y \in K$ of the sequence (x_k). Since F is closed in $K \times \Sigma'$, it is clear that $(y, \sigma) \in F$, and so $y \in F$.

DEFINITION. A compact Hausdorff space K is called *Gul'ko-compact* if the Banach space $C(K)$ of real-valued continuous functions on K, with supremum norm, in its weak topology (induced by its dual space of regular Borel finite measures on K) is $\mathcal{K}$-countably determined.

It is a deep result, due to Vašák [14] (implicitly) and Gul'ko [7] that every Gul'ko-compact space is Corson-compact. (The first result of this type is the fundamental result by Amir and Lindenstrauss [1]: Every *Eberlein-compact* space (i.e., every weakly compact subset of a Banach space) is Corson-compact.)

THEOREM. *If K is a Gul'ko-compact c.c.c. space, then K is metrizable.*

PROOF. By the result of Vašák [14] (implicitly) and Gul'ko [7], every Gul'ko-compact space is Corson-compact. Suppose that there is a Gul'ko-compact c.c.c. nonmetrizable space K. Since the continuous image of a Gul'ko-compact space is (obviously) Gul'ko-compact, we may assume that the weight of K is exactly $\aleph_1$. Thus, $K \subset \Sigma[0,1]^{\omega_1}$. Furthermore, we assume without loss of generality that $\pi_\xi | K \neq 0$ for all $\xi < \omega_1$, and in fact that there is θ, $0 < \theta < 1$, such that

$$V_\xi = (\pi_\xi | K)^{-1}(\theta, 1] \neq \varnothing \quad \text{for } \xi < \omega_1$$

(where $\pi_\xi: [0,1]^{\omega_1} \to [0,1]$ denotes the ξth projection).

Let X be the unit ball of $C(K)$, i.e., $X = \{f \in C(K): \|f\| \leq 1\}$. Since $C(K)$ is weakly $\mathcal{K}$-countably determined, X is too, and so there is a family $\{A_s: s \in S\}$ of subsets of X satisfying the properties of the Lemma. We set $\Pi = \{\pi_\xi: \xi < \omega_1\} \subset X = A_\varnothing$, and $B_s = \{\xi < \omega_1: \pi_\xi \in A_s\}$ for $s \in S$.

For every $s, t \in S$ with $|s| = |t|$ we define $x^t_s, \xi^t_s, V^t_s, C^t_s$ such that:

(i) for every $m < \omega$,

$$\bigcup_{p<\omega} V^{\widehat{t,p}}_{\widehat{s,m}} \quad \text{is dense in } \bigcup \left\{ V_\xi \cap V^t_x : \xi \in C^t_s \cap B_{\widehat{s,m}} \right\};$$

(ii) V^t_s is open in K, $V^{t'}_{s'} \subset V^t_s$ for $t' > t$, $s' > s$, and $x^t_x \in V^t_s$ if $V^t_s \neq \varnothing$;

(iii) $C^t_s \subset B_s$, $C^t_s \cap \operatorname{supp}(x^t_s) = \varnothing$, $C^{t'}_{s'} \subset C^t_s$ for $t' > t$, $s' > s$, $C^{\varnothing}_{\varnothing} = \omega^+$,

$$\bigcup_{p,m<\omega} C^{\widehat{t,p}}_{\widehat{s,m}} \supset C^t_s \setminus \bigcup \left\{ \operatorname{supp}\left(x^{t'}_{s'}\right) : |t'| = |s'| \leqslant |t| + 1 \right\},$$

and if $\xi \in C^t_s$, then $V_\xi \cap V^t_s \neq \varnothing$;

(iv) if $x \in V^t_s$, then $\pi_{\xi^t_s}(x) > \theta$; and,

(v) $\xi^{\widehat{t,p}}_{\widehat{s,m}} \in C^t_s \cap B_{\widehat{s,m}}$ if $C^t_s \cap B_{\widehat{s,m}} \neq \varnothing$.

We proceed inductively; assume that $n < \omega$, and that for every $s, t \in S$, with $|s| = |t| \leqslant n$, $x^t_x, \xi^t_x, V^t_x, C^t_s$ have been defined and they satisfy (i) through (v). Let $t, s \in S$, with $|t| = |s| = n$.

Since K satisfies c.c.c. there are $\xi^{\widehat{t,p}}_{\widehat{s,m}} \in C^t_s \cap B_{\widehat{s,m}}$ for $p < \omega$ and $m < \omega$, such that

$$\bigcup_{p<\omega} \left(V_{\xi^{\widehat{t,p}}_{\widehat{s,m}}} \cap V^t_s \right)$$

is dense in $\bigcup \{ V_\xi \cap V^t_s : \xi \in C^t_s \cap B_{\widehat{s,m}} \}$.

We set

$$V^{\widehat{t,p}}_{\widehat{s,m}} = V^t_s \cap V_{\xi^{\widehat{t,p}}_{\widehat{s,m}}} \quad \text{for } p, m < \omega.$$

If $V^{\widehat{t,p}}_{\widehat{s,m}} \neq \varnothing$, we choose $x^{\widehat{t,p}}_{\widehat{s,m}} \in V^{\widehat{t,p}}_{\widehat{s,m}}$ (and otherwise, we choose $x^{\widehat{t,p}}_{\widehat{s,m}} \in K$, arbitrarily). We set (for $p, m < \omega$)

$$C^{\widehat{t,p}}_{\widehat{s,m}} = \left\{ \xi \in C^t_s \cap B_{\widehat{s,m}} \setminus \operatorname{supp}\left(x^{\widehat{t,p}}_{\widehat{s,m}}\right) : V^{\widehat{t,p}}_{\widehat{s,m}} \cap V_\xi \neq \varnothing \right\}.$$

We note that if $x \in V^{\widehat{t,p}}_{\widehat{s,m}}$ then

$$\pi_{\xi^{\widehat{t,p}}_{\widehat{s,m}}}(x) > \theta, \quad \text{since } V^{\widehat{t,p}}_{\widehat{s,m}} \subset V_{\xi^{\widehat{t,p}}_{\widehat{s,m}}}.$$

The recursive definitions are complete; it is clear that they satisfy properties (i) through (v).

We choose $\xi \in \omega_1 \setminus \bigcup \{\operatorname{supp}(x^t_s) : t, s \in S \,|\, |t| = |s|\}$. By the conditions of the Lemma for the family $\{A_s : s \in S\}$, there is $\sigma \in \Sigma$, such that $\pi_\xi \in \bigcap_{k<\omega} A_{\sigma|k}$, and if $f_k \in A_{\sigma|k}$ for $k < \omega$, then the sequence (f_k) has a weak limit point in X. From property (iii), we choose inductively natural numbers $t_0, \ldots, t_k, \ldots$ for $k < \omega$, such that

$$\xi \in C^{t_0, \ldots, t_k}_{\sigma(0), \ldots, \sigma(k)} \qquad \text{for } k < \omega.$$

Setting $\tau = (t_0, t_1, \ldots, t_k, \ldots) \in \Sigma$, we have that $\xi \in \bigcap_{k<\omega} C^{\tau|k}_{\sigma|k}$.

By property (iii), $V^{\tau|k}_{\sigma|k} \neq \varnothing$ for $k < \omega$; and hence by property (ii), $x^{\tau|k}_{\sigma|k} \in V^{\tau|k}_{\sigma|k}$ for $n < \omega$. Since K is Corson-compact, it is clear (as it was mentioned in the introduction) that there is $x \in K$, and a subsequence of $(x^{\tau|k}_{\sigma|k})$ (which for convenience we denote also by $(x^{\tau|k}_{\sigma|k})$), such that $x^{\tau|k}_{\sigma|k} \to x$.

By property (v), we have that $\pi_{\xi^{\tau|k}_{\sigma|k}} \in A_{\sigma|k}$ for $k < \omega$. By the conditions of the Lemma for the family $\{A_s : s \in S\}$ and Eberlein's theorem for weak convergence, there is $g \in X$, and a subsequence of $(\pi_{\xi^{\tau|k}_{\sigma|k}})$ which for convenience we denote also by $(\pi_{\xi^{\tau|k}_{\sigma|k}})$), such that

$$\pi_{\xi^{\tau|k}_{\sigma|k}} \to g \quad \text{pointwise.}$$

We note that $g \in X \subset C(K)$, i.e., that g is a continuous function. From property (iv), we have that

$$\pi_{\xi^{\tau|l}_{\sigma|l}}\left(x^{\tau|k}_{\sigma|k}\right) > \theta \quad \text{for all } l > k,$$

hence, by the pointwise convergence, $g(x^{\tau|k}_{\sigma|k}) \geqslant \theta$ for all $k < \omega$, and hence by continuity of g, $g(x) \geqslant \theta > 0$. From properties (iii) and (v), we have that

$$\pi_{\xi^{\tau|l}_{\sigma|l}}\left(x^{\tau|k}_{\sigma|k}\right) = 0 \quad \text{for all } l < k,$$

hence by continuity of the projections, and convergence of $x^{\tau|k}_{\sigma|k} \to x$, we have that $\pi_{\xi^{\tau|l}_{\sigma|l}}(x) = 0$ for all $l < \omega$; hence by the pointwise convergence, we have that $g(x) = 0$. This contradiction completes the proof.

The following more general result is proved with the same method.

THEOREM. *If K is a Gul'ko-compact space, then the Souslin number of K is equal to the successor of the weight of K (where the Souslin number of K is the least cardinal α such that there do not exist in K α many nonempty pairwise disjoint open sets).*

We remark that if the weight of K is greater than the power of the continuum, then the above theorem has been proved by Talagrand [**13**, Proposition 7.11].

REMARKS. (1) Rosenthal [**8**] proved that every Eberlein-compact c.c.c. space is metrizable. R. Pol proved that every Gul'ko-compact space satisfying the (stronger than c.c.c.) property of possessing a strictly positive probability Borel measure is metrizable (cf. [**13**, Théorème 6.6]). Call a space K *Talagrand-compact* if $C(K)$ in its weak topology is $\mathfrak{K}$-analytic [**11**]. Talagrand proved in [**10**] that every Eberlein-compact space is Talagrand-compact (but not conversely: [**11**, **13**]). Talagrand in [**12**, Problème 5] has asked if every Talagrand-compact c.c.c. space is metrizable. In [**13**, Problème 7.9] he asks the more general question: Is every Gul'ko-compact c.c.c. space metrizable.

(2) The analogue of the above theorem does not hold for the wider class of Corson-compact spaces, i.e., it cannot be proved that every Corson-compact c.c.c. space is metrizable. A wide variety of counterexamples, assuming the continuum hypothesis, appears in [**3**].

COROLLARY. *Let X be a Banach space, such that X is weakly $\mathfrak{K}$-countably determined, and such that every weakly compact subset of X is separable. In addition, assume that X is isomorphic to a complementary subspace of $C(\Omega)$ for some compact space Ω. Then X is separable.*

Proof. Let X be embedded as a complementary subspace of $C(\Omega)$, and let P: $C(\Omega) \to X$ be the associated projection operator. For $\omega, \omega' \in \Omega$, set $\omega \sim \omega'$ if $f(\omega) = f(\omega')$ for all $f \in X$. Denote by Ω' the quotient space $\Omega/\sim$. Then X is isomorphically embedded in $C(\Omega')$, and $P \mid C(\Omega')$ is the associated projection. Furthermore X separates points of Ω', and thus by results in **[13]**, Ω' is Gul'ko-compact. Thus, without loss of generality, we may assume that Ω is Gul'ko-compact.

Let P^*: $X^* \to M(\Omega)$ be the dual operator (where X^* denotes the dual of X, and $M(\Omega)$ the space of finite regular Borel measures on X, the dual of $C(\Omega)$). For $\mu \in M(\Omega)$, we denote by $\operatorname{supp}(\mu)$, the support of μ, i.e., the closed subset of Ω on which μ is strictly positive. Set

$$K = \text{the closure in } \Omega \text{ of } \bigcup \left\{\operatorname{supp}(P^*(x^*): x^* \in X^*)\right\}.$$

Thus $K \subset \Omega$, and it is easy to see that $X \to C(\Omega) \to C(K)$ is an embedding, and that there is a projection P_1: $C(K) \to X$ such that P is the composition $C(\Omega) \to C(K) \overset{P_1}{\to} X$ (where $C(\Omega) \to C(K)$ is the operator induced by the inclusion $K \subset \Omega$). We claim that K is a c.c.c. space. In fact, suppose that $\{V_\xi: \xi < \omega_1\}$ is a family of pairwise disjoint nonempty open subsets of K. For every $\xi < \omega_1$, there is $x_\xi^* \in X^*$ such that

$$|P_1^*(x_\xi^*)|(V_\xi) > 0.$$

There is $f_\xi \in C(K)$, with $\operatorname{supp}(f_\xi) \subset V_\xi$, such that $\int f_\xi \, d(P_1^*(x_\xi^*)) > 0$, i.e., $P_1(f_\xi) \neq 0$. It follows that the uncountable set $\{P_1(f_\xi): \xi < \omega_1\}$ is discrete in X, and weakly relatively compact, contradicting our assumption that every weakly compact subset of X is separable.

Thus K is a Gul'ko-compact, c.c.c. space, hence by our theorem, K is metrizable, hence $C(K)$ is separable. Thus X is separable.

Remark. The analogue of the above Corollary does not hold for general Banach spaces (as opposed to Banach spaces that are complemented in spaces of continuous functions). I.e., it cannot be proved that if X is a Banach space, such that X is weakly $\mathcal{K}$-countably determined and every weakly compact subset of X is separable, then X is (norm) separable. This statement does hold (for the $\mathcal{K}$-analytic case), if we make an additional set-theoretic assumption, namely Martin's axiom plus the negation of the continuum hypothesis, by a result due to Fremlin **[6]**. It fails, under the assumption of the continuum hypothesis, by an example due to Rosenthal **[9]**, employing a Lusin-set type construction.

Bibliography

1. D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. (2) **88** (1968), 35–46.

2. A. V. Archangel'skii, *Some topological spaces that arise in functional analysis*, Uspehi Mat. Nauk **31** (1976), 17–32 [= Russian Math. Surveys **31** (1976), 14–30].

3. S. Argyros, S. Mercourakis and S. Negrepontis, *Analytic properties of Corson-compact spaces* (Proc. Fifth Prague Topological Sympos., 1981) (to appear).

4. G. Choquet, *Theory of capacities*, Ann. Inst. Fourier **5** (1953), 131–297.

5. W. W. Comfort and S. Negrepontis, *Chain conditions in topology*, Cambridge Univ. Tracts in Math., vol. 79, Cambridge Univ. Press, London and New York, 1982.

6. D. H. Fremlin, *$\mathcal{K}$-analytic spaces with metrizable compacts*, Manuscript, Colchester, 1977.

7. S. P. Gul'ko, *On the structure of spaces of continuous functions and their complete paracompactness*, Uspehi Mat. Nauk **34** (1979), 33–40 [= Russian Math. Surveys **34** (1979), 36–44].

8. H. P. Rosenthal, *On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures μ*, Acta Math. **124** (1970), 205–248.

9. ______, *The heredity problem for weakly compactly generated Banach spaces*, Compostio Math. **28** (1974), 83–111.

10. M. Talagrand, *Sur une conjecture de H. H. Corson*, Bull. Sci. Math. (2) **99** (1975), 211–212.

11. ______, *Espaces de Banach faiblement $\mathcal{K}$-analytiques*, C. R. Acad. Sci. Paris Ser. A–B **284** (1977), A745–A748.

12. ______, *Sur les espaces de Banach faiblement $\mathcal{K}$-analytiques*, C. R. Acad. Sci. Paris. Ser. A–B **285** (1977), A119–A122.

13. ______, *Espaces de Banach faiblement $\mathcal{K}$-analytiques*, Ann. of Math. (2) **110** (1979), 407–438.

14. L. Vašák, *On one generalization of weakly compactly generated Banach spaces*, Studia Math. **70** (1980), 11–19.

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