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ON THE NORMAL STRUCTURE COEFFICIENT AND THE BOUNDED SEQUENCE COEFFICIENT

TECK-CHEONG LIM

ABSTRACT. The two notions of normal structure coefficient and bounded sequence coefficient introduced by Bynum are shown to be the same. A lower bound for the normal structure coefficient in L^p, $p > 2$, is also given.

Let X be a Banach space and C a closed convex bounded subset of X. For each x in C, let $R(x, C) = \sup\{\|x - y\|: y \text{ in } C\}$ and let $R(C)$ denote the Chebyshev radius of C [**2**, p. 178]:

$$R(C) = \inf\{R(x, C): x \text{ in } C\}.$$

Let $D(C)$ denote the diameter of C, $D(C) = \sup\{\|x - y\|: x, y \in C\}$.

For a bounded sequence $\{x_n\}$ in X, the asymptotic diameter $A(\{x_n\})$ of $\{x_n\}$ is defined to be $\lim_n \sup\{\|x_k - x_m\|: m \geqslant n, k \geqslant n\}$.

In [**1**], Bynum introduced the following two coefficients of X, called the normal structure coefficient and the bounded sequence coefficient respectively:

$$N(X) = \inf\{D(C)/R(C): C \text{ closed convex bounded nonempty subsets of } X \text{ with } |C| > 1\},$$

$$BS(X) = \sup\Big\{M: \text{for every bounded sequence } \{x_n\} \text{ in } X, \text{ there exists } y \text{ in } \overline{\mathrm{Co}}\,(x_n) \text{ such that } M \limsup_n \|x_n - y\| \leqslant A(\{x_n\})\Big\}.$$

Another coefficient relating to the asymptotic radius of a sequence (see e.g. [**3**]) can be defined as follows: Let $\{x_n\}$ be a bounded sequence in X. For each x in X, define

$$r(x, \{x_n\}) = \limsup_n \|x_n - x\|.$$

The number $r(\{x_n\}) = \inf\{f(x, \{x_n\}): x \in \overline{\mathrm{Co}}(x_n)\}$ will be called the asymptotic radius of $\{x_n\}$, or more precisely, the asymptotic radius of $\{x_n\}$ w.r.t. $\overline{\mathrm{Co}}(x_n)$. We shall denote the coefficient

$$\inf\{A(\{x_n\})/r(\{x_n\}): \{x_n\} \text{ bounded nonconvergent sequences in } X\}$$

by $A(X)$.

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In [1], Bynum mentioned that the two coefficients $N(X)$ and $BS(X)$ are equal in a separable Banach space X. In this note, we shall show that the three coefficients are equal in any Banach space X.

THEOREM 1. *For a Banach space* X, $N(X) = BS(X) = A(X)$.

PROOF. It follows readily from the definition that $BS(X) = A(X)$. Indeed, we may assume that the sequences in the definition of $BS(X)$ are nonconvergent. Clearly $BS(X) \leqslant A(X)$. On the other hand for each $\lambda > 1$, $A(X)/\lambda$ belongs to the defining set of $BS(X)$ and thus $BS(X) \geqslant A(X)$. Bynum [1] proved that $N(X) \leqslant BS(X)$. To prove that $BS(X) \leqslant N(X)$, it suffices to show that for any bounded convex nonempty set C with more than one point, there is a separable closed convex subset C_1 such that $R(C_1) = R(C)$. Indeed, if $\{x_n\}$ is a dense sequence in C_1 and M is a number in the defining set of $BS(X)$, then

$$M \leqslant A(\{x_n\})/\limsup_n \|x_n - y\| = \frac{D(C_1)}{R(y, C_1)} \leqslant \frac{D(C_1)}{R(C_1)} \leqslant \frac{D(C)}{R(C)}.$$

To construct C_1, we start out with a sequence of points $\{z_n\}$ in C such that $\lim_{n\to\infty} R(z_n, C) = R(C)$. Let $U_1 = \mathrm{Co}(\{z_n\})$. Let $V_1 = \{x \in U_1 : R(x, U_1) < R(C)\}$ and let W_1 be a countable dense subset of V_1. For each x in W_1, let D_x be a sequence of points in C such that $R(x, D_x) \geqslant R(C)$. Let X_1 be the countable set $\bigcup\{D_x : x \in W_1\}$ and $U_2 = \mathrm{Co}(U_1 \cup X_1)$. We define similarly V_2, W_2 and X_2 from U_2 and continue this process to obtain an increasing sequence of convex sets $U_1 \subset U_2 \subset U_3 \subset \cdots \subset U_n \subset \cdots$. Let $C_1 = \overline{\mathrm{Co}}(\bigcup U_n)$. C_1 is separable. Since $R(z_n, C_1) \leqslant R(z_n, C)$ and $\lim_{n\to\infty} R(z_n, C) = R(C)$, we have $R(C_1) \leqslant R(C)$. From the way U_n are constructed, $R(x, U_{n+1}) \geqslant R(C)$ for each $x \in U_n$. It follows that $R(C_1) \geqslant R(C)$ and the proof is complete.

For $0 < \mu \leqslant \frac{1}{2}$ and $p > 2$, denote by $x(\mu)$ the unique solution of the equation

$$\lambda x^{p-1} - \mu - (\lambda x - \mu)^{p-1} = 0$$

in the interval $\mu/\lambda \leqslant x \leqslant 1$. Define $g(\mu)$, $0 < \mu \leqslant 1$, by

$$g(\mu) = \lambda\mu \frac{1 + x(\lambda \wedge \mu)^{p-1}}{(1 + x(\lambda \wedge \mu))^{p-1}}$$

where $\lambda = 1 - \mu$. We proved in [4] the following inequality in L^p $(p > 2)$:

$$\|\lambda x + \mu y\|^p + g(\mu)\|x - y\|^p \leqslant \lambda\|x\|^p + \mu\|x\|^p$$

and that

$$\sup_{0<\mu\leqslant 1} \frac{g(\mu)}{\mu} = \frac{1 + \alpha^{p-1}}{(1 + \alpha)^{p-1}},$$

where α is the unique solution of

$$(p - 2)x^{p-1} + (p - 1)x^{p-2} - 1 = 0$$

in the interval $0 \leqslant x \leqslant 1$.

THEOREM 2. *For* $X = L^p, p > 2$,

$$N(X) \geqslant \left(1 + \frac{1 + \alpha^{p-1}}{(1+\alpha)^{p-1}}\right)^{1/p}.$$

PROOF. For a closed convex bounded set C in X, let R and D be the Chebyshev radius and the diameter of C respectively. Let z be the Chebyshev center of C. For x, y in C and $0 < \mu \leqslant 1$, we have

$$\|\lambda z + \mu y - x\|^p + g(\mu)\|z - y\|^p \leqslant \lambda\|z - x\|^p + \mu\|y - x\|^p.$$

Taking sup over x in C and noting that $R \leqslant \sup\{\|\lambda z + \mu y - x\| \colon x \in C\}$, we obtain

$$R^p + g(\mu)\|z - y\|^p \leqslant \lambda R^p + \mu \sup\{\|y - x\|^p \colon x \in C\}.$$

It follows, after taking sup over y in C, that $(\mu + g(\mu))R^p \leqslant \mu D^p$ and hence

$$\frac{D}{R} \geqslant \left(1 + \sup_{0<\mu\leqslant 1} \frac{g(\mu)}{\mu}\right)^{1/p} = \left(1 + \frac{1 + \alpha^{p-1}}{(1+\alpha)^{p-1}}\right)^{1/p}.$$

Therefore

$$N(X) \geqslant \left(1 + \frac{1 + \alpha^{p-1}}{(1+\alpha)^{p-1}}\right)^{1/p}. \quad \square$$

REMARK 1. For $p = 3$ and 4, we have $\alpha = \sqrt{2} - 1$ and $1/2$ and hence

$$\left(1 + \frac{1 + \alpha^{p-1}}{(1+\alpha)^{p-1}}\right)^{1/p} = \left(3 - \sqrt{2}\right)^{1/3} \text{ and } (4/3)^{1/4}$$

respectively.

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DEPARTMENT OF MATHEMATICS, GEORGE MASON UNIVERSITY, 4400 UNIVERSITY DRIVE, FAIRFAX, VIRGINIA 22030