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TENSOR PRODUCTS OF PRECLOSED OPERATORS ON C^*-ALGEBRAS

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ABSTRACT. In this paper, we prove the following result: If A_1, A_2 are C^*-algebras, and T_1, T_2 are preclosed operators on A_1, A_2 respectively, then $T_1 \otimes T_2$ is preclosed on $A_1 \otimes_{\min} A_2$. Furthermore, we show that the injective C^*-cross norm $\|\cdot\|_{\min}$ is reflexive on the algebraic tensor product $A_1 \otimes A_2$.

Since Turumaru [**8**] introduced tensor products of C^*-algebras, mysterious properties of C^*-cross norms received much attention from many specialists. For example, Takesaki [**6**] found out that the C^*-norm on the algebraic tensor product is not unique. Furthermore, Okayasu [**3**] showed that the minimal C^*-norm is not a uniform cross norm. Therefore, it is natural to ask the question: when σ and τ are bounded operators on C^*-algebras A_1 and A_2, is $\sigma \otimes \tau$ bounded on $A_1 \otimes_{\min} A_2$? The answer is negative. In 1970, Okayasu [**3**] gave an example of σ, τ and C^*-algebras A_1, A_2 in such a way that $\sigma \otimes \tau$ is unbounded on $A_1 \otimes_{\min} A_2$. Naturally, we will ask the question: what can we say about $\sigma \otimes \tau$ when σ and τ are bounded? What kind of properties of the operators σ and τ are preserved under the tensor product operation?

The purpose of this paper is to answer these questions. (1) If σ and τ are bounded, then $\sigma \otimes \tau$ is preclosed on $A_1 \otimes_{\min} A_2$. (2) If σ and τ are densely defined and preclosed, then $\sigma \otimes \tau$ is densely defined and preclosed on $A_1 \otimes_{\min} A_2$.

In Theorem 3, we show that the minimal C^*-cross norm is reflexive.

LEMMA 1. *Let E be a Banach space and T be a densely defined linear operator on E. Then the following statements are equivalent.*

(1) *T is preclosed.*

(2) *T has a minimal closed linear extension*; *i.e., there exists a closed linear extension $\bar{T}$ of T such that any closed linear extension of T is a closed linear extension of $\bar{T}$.*

(3) *For any $y \neq 0$ in E, $(0, y)$ is not in the closure of the graph of T.*

(4) *$\mathfrak{D}(T^*)$ is total in E^*.*

(5) *$\mathfrak{D}(T^*)$ is $\sigma(E^*, E)$-dense in E^*.*

(*T^* is the conjugate operator of T*; *E^* is the conjugate space of E.*)

Since we can find the proof of Lemma 1 in general Banach space text books, we omit it.

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LEMMA 2. *Let A_1, A_2 be C^*-algebras and A_1^* and A_2^* be the conjugate spaces of A_1, A_2, respectively. Then the algebraic tensor product*

$$F = A_1^* \otimes A_2^*$$

is $\sigma((A_1 \otimes_{\min} A_2)^*, A_1 \otimes_{\min} A_2)$-dense in $(A_1 \otimes_{\min} A_2)^*$.

PROOF. From p. 208 in [7], if $f_1 \in A_1^*, f_2 \in A_2^*$ and $x = \sum_{i=1}^n x_{1,i} \otimes x_{2,i} \in A_1 \otimes A_2$, then

$$|\langle x, f_1 \otimes f_2 \rangle| \leqslant \|x\|_{\min} \|f_1\| \|f_2\|.$$

Thus, we have $A_1^* \otimes A_2^* \subseteq (A_1 \otimes_{\min} A_2)^*$.

To be $\sigma((A_1 \otimes_{\min} A_2)^*, A_1 \otimes_{\min} A_2)$-dense in $(A_1 \otimes_{\min} A_2)^*$ is equivalent to being total in $(A_1 \otimes_{\min} A_2)^*$ so we have to prove that F is total in $(A_1 \otimes_{\min} A_2)^*$.

Suppose $x \in A_1 \otimes_{\min} A_2$ and $\langle x, f \rangle = 0, \forall f \in F$. We shall show $x = 0$.

Let $\sigma(A_1)$ and $\sigma(A_2)$ be the state spaces of A_1 and A_2 respectively, $\omega_1 \in \sigma(A_1)$ and $\omega_2 \in \sigma(A_2)$.

Let Π_{ω_1} and Π_{ω_2} be the cyclic representations corresponding to ω_1 and ω_2 with representation Hilbert spaces $\mathfrak{G}_{\omega_1}$ and $\mathfrak{G}_{\omega_2}$ respectively. We construct the tensor representation of Π_{ω_1} and Π_{ω_2},

$$\Pi_\omega = \Pi_{\omega_1} \otimes \Pi_{\omega_2}.$$

Therefore, for all $\xi_1, \eta_1 \in \mathfrak{G}_{\omega_1}$ and $\xi_2, \eta_2 \in \mathfrak{G}_{\omega_2}$,

$$(\Pi_\omega(x)(\xi_1 \otimes \xi_2) \mid \eta_1 \otimes \eta_2) = \langle x, f \otimes g \rangle = 0,$$

in which $\langle x_1, f \rangle = (\Pi_{\omega_1}(x_1)\xi_1 \mid \eta_1)$ and $\langle x_2, g \rangle = (\Pi_{\omega_2}(x_2)\xi_2 \mid \eta_2)$. Then

$$\left(\Pi_\omega(x)(\xi_1 \otimes \xi_2) \mid \sum_{j=1}^m \eta_{1,j} \otimes \eta_{2,j}\right) = 0.$$

Hence $\Pi_\omega(x)(\xi_1 \otimes \xi_2) \perp \mathfrak{G}_{\omega_1} \otimes \mathfrak{G}_{\omega_2}$.

Since $\mathfrak{G}_{\omega_1} \otimes \mathfrak{G}_{\omega_2}$ is dense in $\mathfrak{G}_\omega$, $\Pi_\omega(x)(\xi_1 \otimes \xi_2) = 0$; also

$$\Pi_\omega(x)\left(\sum_{i=1}^n \xi_{1,i} \otimes \xi_{2,i}\right) = 0, \qquad \Pi_\omega(x) = 0.$$

This implies

$$\|x\|_{\min} = \operatorname{Sup}\left\{\|\Pi_\omega(x)\| : \omega = \omega_1 \otimes \omega_2, \begin{matrix} \omega_1 \in \sigma(A_1) \\ \omega_2 \in \sigma(A_2) \end{matrix}\right\} = 0.$$

Thus we have $x = 0$. Q.E.D.

THEOREM 1. *Let A_1 and A_2 be C^*-algebras and T_1 and T_2 be densely defined preclosed operators on A_1 and A_2. Then $T_1 \otimes T_2$ is preclosed on $A_1 \otimes_{\min} A_2$.*

PROOF. From Lemma 1, as T_1 and T_2 are densely defined preclosed operators on A_1 and A_2, it follows that $\mathfrak{D}(T_1^*)$ is $\sigma(A_1^*, A_1)$-dense in A_1^* and $\mathfrak{D}(T_2^*)$ is $\sigma(A_2^*, A_2)$-dense in A_2^*.

It is easy to verify $\mathfrak{D}(T_1^*) \otimes \mathfrak{D}(T_2^*)$ is $\sigma((A_1 \otimes_{\min} A_2)^*, (A_1 \otimes_{\min} A))$-dense in $A_1^* \otimes A_2^*$.

By Lemma 2, we can conclude

$$\mathfrak{D}(T_1^*) \otimes \mathfrak{D}(T_2^*)$$

is $\sigma((A_1 \otimes_{\min} A_2)^*, (A_1 \otimes_{\min} A_2))$-dense in $(A_1 \otimes_{\min} A_2)^*$.

Since

$$\mathfrak{D}(T_1^*) \otimes \mathfrak{D}(T_2^*) \subseteq \mathfrak{D}((T_1 \otimes T_2)^*),$$

$\mathfrak{D}((T_1 \otimes T_2)^*)$ is $\sigma((A_1 \otimes_{\min} A_2)^*, (A_1 \otimes_{\min} A_2))$-dense in $(A_1 \otimes_{\min} A_2)^*$. By Lemma 1, $T_1 \otimes T_2$ is preclosed in $(A_1 \otimes_{\min} A_2)$. Q.E.D.

COROLLARY 1. *Let T_1, T_2 be bounded operators on A_1 and A_2, respectively. Then $T_1 \otimes T_2$ is preclosed on $A_1 \otimes_{\min} A_2$.*

Now we turn to studying tensor products of Banach spaces. We will use the notations in [7].

We assume that E_1 and E_2 denote any two Banach spaces while E_1^* and E_2^* stand for their conjugate spaces, $E_1 \otimes E_2$ and $E_1^* \otimes E_2^*$ denote algebraic tensor products of E_1, E_2 and E_1^*, E_2^* respectively.

If β is a norm in $E_1 \otimes E_2$, then β induces naturally a norm on $E_1^* \otimes E_2^*$:

$$\| f \|_{\beta^*} = \operatorname{Sup}\{ | \langle x, f \rangle | : x \in E_1 \otimes E_2, \| x \|_\beta \leq 1 \},$$

in which $f = \sum_{i=1}^n f_{1,i} \otimes f_{2,i} \in E_1^* \otimes E_2^*$, where $\langle x, f \rangle$ means, of course, the value

$$\langle x, f \rangle = \sum_{j=1}^{m} \sum_{i=1}^{n} \langle x_{1,j}, f_{1,i} \rangle \langle x_{2,j}, f_{2,i} \rangle,$$

for each $x = \sum_{j=1}^m x_{1,j} \otimes x_{2,j} \in E_1 \otimes E_2$.

In the same way, we can define β^{**}.

If $\beta^{**} = \beta$ on $E_1 \otimes E_2$, we call β reflexive.

The completion of $E_1 \otimes E_2$ and $E_1^* \otimes E_2^*$ under β and β^* are denoted by $E_1 \otimes_\beta E_2$ and $E_1^* \otimes_{\beta^*} E_2^*$.

We suppose λ is the least norm on $E_1 \otimes E_2$ [7].

THEOREM 2. *If β is a reflexive norm on $E_1 \otimes E_2$, $\beta \geq \lambda$ and T_1, T_2 are densely defined preclosed operators on E_1 and E_2 respectively, then $T_1 \otimes T_2$ is preclosed on $E_1 \otimes_\beta E_2$.*

PROOF. As T_1 and T_2 are preclosed, by Lemma 1, $\mathfrak{D}(T_1^*)$ is $\sigma(E_1^*, E_1)$-dense in E_1^* and $\mathfrak{D}(T_2^*)$ is $\sigma(E_2^*, E_2)$-dense in E_2^* respectively.

Now we prove $E_1^* \otimes_{\beta^*} E_2^*$ is $\sigma((E_1 \otimes_\beta E_2)^*, (E_1 \otimes_\beta E_2))$-dense in $(E_1 \otimes_\beta E_2)^*$. Equivalently, we have to prove $E_1^* \otimes_{\beta^*} E_2^*$ is total in $(E_1 \otimes_\beta E_2)^*$.

Since $\beta \geq \lambda$, from [4] $E_1^* \otimes E_2^* \subseteq (E_1 \otimes_\beta E_2)^*$ and $E_1^* \otimes_{\beta^*} E_2^* \subseteq (E_1 \otimes_\beta E_2)^*$. As β is reflexive, by Lemma 4.1 of [5], $(E_1 \otimes_\beta E_2) \subseteq (E_1^{**} \otimes_{\beta^{**}} E_2^{**})$.

Let x be an element in $E_1 \otimes_\beta E_2$ such that $f(x) = 0$ for all $f \in E_1^* \otimes_{\beta^*} E_2^*$.

By continuity of β^* and definition, $\beta^{**}(x)$ is the least positive number for which $|f(x)| \leq C\beta^*(f)$, for all $f \in E_1^* \otimes_{\beta^*} E_2^*$.

Therefore, $\beta^{**}(x) = 0$.

According to assumption, $\beta(x) = \beta^{**}(x) = 0$ and $x = 0$.

Hence, $E_1^* \otimes E_2^*$ is $\sigma((E_1 \otimes_\beta E_2)^*, (E_1 \otimes_\beta E_2))$-dense in $(E_1 \otimes_\beta E_2)^*$. Therefore, $\mathfrak{D}(T_1^*) \otimes \mathfrak{D}(T_2^*)$ is $\sigma((E_1 \otimes_\beta E_2)^*, E_1 \otimes_\beta E_2)$-dense in $(E_1 \otimes_\beta E_2)^*$.

Since

$$\mathfrak{D}(T_1^*) \otimes \mathfrak{D}(T_2^*) \subseteq \mathfrak{D}((T_1 \otimes T_2)^*),$$

$\mathfrak{D}((T_1 \otimes T_2)^*)$ is $\sigma((E_1 \otimes_\beta E_2)^*, (E_1 \otimes_\beta E_2))$-dense in $(E_1 \otimes_\beta E_2)^*$.

By Lemma 1, $T_1 \otimes T_2$ is preclosed. Q.E.D.

Now we prove that the injective C^*-cross norm is reflexive. Using this property, we can give another proof of Theorem 1.

THEOREM 3. *If A_1 and A_2 are C^*-algebras, the injective C^*-cross norm on the algebraic tensor product $A_1 \otimes A_2$ is reflexive.*

PROOF. Let

$$A = A_1 \otimes_\alpha A_2, \qquad V = A_1^* \otimes_{\alpha^*} A_2^*.$$

S_{α^*}, S_α denote the unit ball of V, A respectively, that is,

$$S_{\alpha^*} = \{\omega \mid \|\omega\|_{\alpha^*} \leq 1, \omega \in V\},$$
$$S_\alpha = \{a \mid \|a\|_\alpha \leq 1, a \in A\}.$$

We further set

$$\langle x, \omega a \rangle = \langle ax, \omega \rangle, \quad \langle x, a\omega \rangle = \langle xa, \omega \rangle.$$

Since $\|a\omega\|_{\alpha^*} \leq \|a\|_\alpha \|\omega\|_{\alpha^*}$ for $a \in A_1 \otimes_\alpha A_2$ and $\omega \in V$, V is invariant under A. That is, if $\omega \in V$, then $\omega a \in V$, $a\omega \in V$ for all $a \in A$.

For $a \in A$, we define $|||a||| = \operatorname{Sup}\{\|a\omega\|_{\alpha^*}: \omega \in V, \|\omega\|_{\alpha^*} \leq 1\}$.

It is easy to verify

$$|||a + b||| \leq |||a||| + |||b|||, \quad |||ab||| \leq |||a|||\,|||b|||,$$
$$|||\lambda a||| = |\lambda|\,|||a|||, \quad |||a||| \leq \|a\|_\alpha \quad \text{for } a, b \in A.$$

According to the minimal property of the norm [**1**], we have

$$|||a||| = \|a\|_\alpha.$$

$$\begin{aligned}\|a\|_\alpha &= \operatorname{Sup}\{\|a\omega\|_{\alpha^*}: \omega \in S_{\alpha^*}\}\\ &= \operatorname{Sup}\{|\langle b, a\omega\rangle| : b \in S_\alpha, \omega \in S_{\alpha^*}\}\\ &= \operatorname{Sup}\{|\langle ba, \omega\rangle| : b \in S_\alpha, \omega \in S_{\alpha^*}\}\\ &= \operatorname{Sup}\{|\langle a, \omega b\rangle| : b \in S_\alpha, \omega \in S_{\alpha^*}\}\\ &\leq \operatorname{Sup}\{|\langle a, \omega\rangle| : \omega \in S_{\alpha^*}\} \leq \|a\|_\alpha.\end{aligned}$$

Therefore, $\|a\|_\alpha = \|a\|_{\alpha^{**}}$. Q.E.D.

COROLLARY 2. *Let A_1 and A_2 be C^*-algebras and T_1 and T_2 be densely defined preclosed operators on A_1 and A_2. Then $T_1 \otimes T_2$ is preclosed on $A_1 \otimes_{\min} A_2$.*

PROOF. Since $\|\cdot\|_{\min}$ is reflexive and $\lambda \leq \|\cdot\|_{\min}$, by Theorem 2, $T_1 \otimes T_2$ is preclosed on $A_1 \otimes_{\min} A_2$. Q.E.D.

In fact, Corollary 2 gives another proof of Theorem 1.

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References

1. F. Bonsall, *A minimal property of the norm in some Banach algebra*, J. London. Math. Soc. **29** (1954), 156–164.

2. T. Okayasu, *On the tensor products of C^*-algebra*, Tôhoku Math. J. **18** (1966), 325–331.

3. ______, *Some cross norms which are not uniformly cross*, Proc. Japan. Acad. **45** (1970), 54–57.

4. R. Schatten, *A theory of cross-spaces*, Ann. of Math. Studies, no. 26, Princeton Univ. Press, Princeton, N. J., 1950.

5. ______, *On reflexive norms for the direct product*, Trans. Amer. Math. Soc. **54** (1943), 498–506.

6. M. Takesaki, *On the cross-norm of the direct product of C^*-algebra*, Tôhoku Math. J. **16** (1964), 111–122.

7. ______, *Theory of operator algebras*. I, Springer-Verlag, New York, 1979.

8. T. Turumaru, *On the direct product of operator algebra*. I, Tôhoku Math. J. **4** (1952), 242–251.

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