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COEFFICIENTS AND INTEGRAL MEANS OF SOME CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. The sharp coefficient bounds for the classes V_k of functions of bounded boundary rotation are obtained by a short and elementary argument. Elementary methods are also applied for the coefficients of related classes characterised by a generalised Kaplan condition. The result $(1+xz)^{\alpha}(1-z)^{-\beta} \ll (1+z)^{\alpha}(1-z)^{-\beta}$ ($|x|=1$, $\alpha \geqslant 1$, $\beta \geqslant 1$) is proved simply. It is further shown that the functions $(1+z)^{\alpha}(1-z)^{-\beta}$ are extremal for the pth means (p an arbitrary real) of all Kaplan classes $K(\alpha, \beta)$.

1. The Kaplan classes. A function $f(z) = 1 + a_1 z + a_2 z^2 + \cdots$ analytic and nonzero in $|z| < 1$ is said to belong to the *Kaplan class* $K(\alpha, \beta)$ ($\alpha \geqslant 0$, $\beta \geqslant 0$) if for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$ we have

$$-\alpha\pi \leqslant \int_{\theta_1}^{\theta_2} \left\{ \operatorname{Re} \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} - \frac{1}{2}(\alpha - \beta) \right\} d\theta \leqslant \beta\pi. \tag{1}$$

Notice that each of these inequalities implies the other. This definition includes several well-known classes.

(i) $g(z) = z + \frac{1}{2}a_1 z^2 + \cdots$ is close-to-convex of order α iff $g' \in K(\alpha, \alpha + 2)$.

(ii) $f \in K(\alpha, \alpha)$ iff for a suitable real μ,

$$\left|\arg\left(e^{i\mu} f(z)\right)\right| \leqslant \alpha\pi/2 \qquad (|z| < 1). \tag{2}$$

(iii) $g(z) = z + \cdots$ is starlike of order $\lambda < 1$ iff $g(z)/z \in K(0, 2(1-\lambda))$.

An alternative definition can be formulated as follows. For λ real we write

$$\Pi_\lambda = \begin{cases} K(\lambda, 0) & (\lambda \geqslant 0), \\ K(0, -\lambda) & (\lambda < 0), \end{cases} \tag{3}$$

or, equivalently, $f \in \Pi_\lambda$ iff for $|z| < 1$,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \begin{cases} < \frac{1}{2}\lambda & (\lambda > 0), \\ > \frac{1}{2}\lambda & (\lambda < 0). \end{cases} \tag{4}$$

The class $\Pi_0 = K(0,0)$ consists of the single function $f(z) = 1$. We then have

THEOREM A [10]. *$f \in K(\alpha, \beta)$ iff we can write $f(z) = g(z)H(z)$, where $g \in \Pi_{\alpha-\beta}$, $|\arg(e^{i\mu}H)| \leqslant \frac{1}{2}\pi \min(\alpha, \beta)$ for a suitable real μ.*

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THEOREM B. (a) $0 \leqslant \alpha' \leqslant \alpha$, $0 \leqslant \beta' \leqslant \beta \Rightarrow K(\alpha', \beta') \subset K(\alpha, \beta)$.
(b) $f \in K(\alpha, \beta) \Leftrightarrow 1/f \in K(\beta, \alpha)$.
(c) $f \in K(\alpha, \beta) \Leftrightarrow$ *for each* $p > 0, f^p \in K(p\alpha, p\beta)$.
(d) $f \in K(\alpha, \beta)$, $g \in K(\alpha', \beta') \Rightarrow fg \in K(\alpha + \alpha', \beta + \beta')$.

The functions in Π_λ are characterized by the representation

$$f(z) = \exp\left(\lambda \int_T \log(1 + e^{it}z)\, d\mu(t)\right) \tag{5}$$

for a suitable probability measure on the unit circle T. This gives as a dense subclass the λ-products

$$f(z) = \prod_{k=1}^{n} (1 + x_k z)^{\lambda_k} \tag{6}$$

where $|x_k| \leqslant 1$, $\operatorname{sign} \lambda_k = \operatorname{sign} \lambda$, $\Sigma_1^n \lambda_k = \lambda$. Of special interest are the classes $S(\alpha, \beta)$, where $\alpha \geqslant 0$, $\beta \geqslant 0$, consisting of functions of the form

$$f(z) = g(z)/h(z) \tag{7}$$

where $g \in \Pi_\alpha$, $h \in \Pi_\beta$. From Theorem B we see that

$$S(\alpha, \beta) \subset K(\alpha, \beta), \tag{8}$$

and if $\alpha > 0$, $\beta > 0$, this containment is strict. It is well known that a function $g(z) = z + a_1 z^2 + \cdots$ has bounded boundary rotation not exceeding $k\pi$ (the class V_k where $k \geqslant 2$) iff $g' \in S(\frac{1}{2}k - 1, \frac{1}{2}k + 1)$. In particular, such functions g are close-to-convex of order $\frac{1}{2}k - 1$ **[4]**.

2. The coefficient problem. The sharp bounds for the coefficients of functions in V_k were obtained over two substantial papers **[1, 4]**. The first of these **[4]** reduced the problem by means of some ingenious extreme point arguments to estimating the coefficients of the special functions $(1 + xz)^\alpha(1 - z)^{-\alpha}$, where $|x| = 1$, $\alpha \geqslant 1$. The estimate

$$\left(\frac{1 + xz}{1 - z}\right)^\alpha \ll \left(\frac{1 + z}{1 - z}\right)^\alpha \qquad (|x| = 1, \alpha \geqslant 1) \tag{9}$$

was obtained with some difficulty in **[1]** and established the conclusion

$$f(z) \ll (1 + z)^\alpha / (1 - z)^{\alpha + 2} \tag{10}$$

for $f \in K(\alpha, \alpha + 2)$ $(\alpha \geqslant 0)$. Later Brannan **[3]** simplified the proof of (9) and some similar results, but considerable ingenuity was still required. Even deeper convolution methods, as well as Brannan's results were required to show that

$$f(z) \ll (1 + z)^\alpha / (1 - z)^\beta \tag{11}$$

for $f \in K(\alpha, \beta)$ $(\alpha \geqslant 1, \beta \geqslant 1)$ **[9, 10]**. There is a gap in these results. It is still true that (11) holds when $0 < \alpha < 1$, $\beta \geqslant 2 - \alpha$. The proof is completely elementary.

THEOREM 1. *If* $f \in K(\alpha, \beta)$, *where* $0 \leqslant \alpha \leqslant 1$, $\beta \geqslant 2 - \alpha$, *then*

$$f(z) \ll (1 + z)^\alpha / (1 - z)^\beta. \tag{12}$$

PROOF. We can write $f = gF$ where $g \in \Pi_{\alpha-\beta}$, $F \in K(\alpha, \alpha)$. Then

$$f(z) = \left(F(z)g(z)^{(\beta-1)/(\beta-\alpha)}g(-z)^{(\alpha-1)/(\beta-\alpha)}\right)\left(g(z)g(-z)\right)^{(1-\alpha)/(\beta-\alpha)}.$$

Now $g(z)^{(\beta-1)/(\beta-\alpha)} \in K(0, \beta - 1)$ and $g(-z)^{(\alpha-1)/(\beta-\alpha)} \in K(1 - \alpha, 0)$. Hence

$$F(z)g(z)^{(\beta-1)/(\beta-\alpha)}g(-z)^{(\alpha-1)/(\beta-\alpha)} \in K(1, \alpha + \beta - 1)$$

and so can be written in the form Hp, where $H \in K(1, 1)$ and $p \in \Pi_{2-\alpha-\beta}$. Standard estimates give $H(z) \ll (1 + z)(1 - z)^{-1}$, $p(z) \ll (1 - z)^{2-\alpha-\beta}$. Thus

$$H(z)p(z) \ll (1 + z)/(1 - z)^{\alpha+\beta-1}. \tag{13}$$

Secondly,

$$(g(z)g(-z))^{(1-\alpha)/(\beta-\alpha)} = k(z^2),$$

where $k(z) \in \Pi_{\alpha-1}$, which implies

$$k(z^2) \ll (1 - z^2)^{\alpha-1}. \tag{14}$$

From (13) and (14) we obtain

$$f(z) \ll \frac{1 + z}{(1 - z)^{\alpha+\beta-1}}(1 - z^2)^{\alpha-1} = \frac{(1 + z)^\alpha}{(1 - z)^\beta}. \tag{15}$$

The solution of the V_k problem is an immediate consequence:

COROLLARY. *If $f \in K(\alpha, \beta)$, where $\beta - \alpha \geqslant 2(1 - \{\alpha\})$, then* (12) *holds. In particular*, (10) *holds.*

PROOF. If $m = [\alpha] + 1 = \alpha + 1 - \{\alpha\}$, we apply the theorem to $f^{1/m} \in K(\alpha/m, \beta/m)$.

With the help of Theorem 1 we obtain a simple proof of the result of Aharonov and Friedland [**1**]; also see Brannan [**3**].

THEOREM 2. *For $\alpha \geqslant 1$, $\beta \geqslant 1$ we have*

$$(1 + xz)^\alpha/(1 - z)^\beta \ll (1 + z)^\alpha/(1 - z)^\beta \qquad (|x| \leqslant 1). \tag{16}$$

PROOF. Since $(1 + xz)^m \ll (1 + z)^m$ for any nonnegative integer m, we may assume that $1 < \alpha < 2$, $\beta = 1$. Put $\alpha = 1 + \gamma$ and consider

$$g(z) = (1 + xz)^{1+\gamma}(1 - z)^{-1}.$$

Differentiating gives

$$g'(z) = \frac{(1 + xz)^\gamma}{(1 - z)^{2-\gamma}} \frac{1 + (\gamma + 1)x - \gamma xz}{(1 - z)^\gamma}.$$

By Theorem 1,

$$(1 + xz)^\gamma/(1 - z)^{2-\gamma} \ll (1 + z)^\gamma/(1 - z)^{2-\gamma}.$$

It remains to prove

$$(1 + (\gamma + 1)x - \gamma xz)/(1 - z)^\gamma \ll (2 + \gamma - \gamma z)/(1 - z)^\gamma, \tag{17}$$

with the right-hand expression having nonnegative coefficients. The left-hand expression is clearly $\ll 1/(1-z)^{\gamma} + (\gamma + 1 - \gamma z)/(1-z)^{\gamma}$ providing that the second term has nonnegative coefficients, which will also show that the right-hand expression in (17) has nonnegative coefficients. The proof is completed by observing that

$$\frac{d}{dz}\left(\frac{1-\gamma z}{(1-z)^{\gamma}}\right) = \frac{\gamma(1-\gamma)z}{(1-z)^{\gamma+1}}$$

has nonnegative coefficients for $0 < \gamma < 1$.

REMARK 1. Although, as we have shown, the coefficient problem for V_k can be solved by elementary methods, nevertheless the extreme point methods introduced in [4] seem to be essential for proving (11) in the general case. In view of Theorem 1 it remains an interesting open problem as to whether the functions $(1 + xz)^{\alpha}(1 - yz)^{-\beta}$ $(|x| = |y| = 1)$ represent the extreme points of $K(\alpha, \beta)$ for $0 < \alpha < 1$, $\beta \geqslant 2 - \alpha$.

The coefficient problem for the remaining values of the parameters α and β presents a number of difficulties. In general the function $(1 + z)^{\alpha}(1 - z)^{-\beta}$ is no longer extremal. The case $\beta = \alpha$ is easily dealt with.

THEOREM 3. *If $f \in K(\alpha, \alpha)$ where $0 < \alpha < 1$, then*

$$|a_n| \leqslant 2\alpha \qquad (n = 1, 2, \ldots). \tag{18}$$

This is sharp for $f(z) = (1 + z^n)^{\alpha}(1 - z^n)^{-\alpha}$.

PROOF. Since $f^{1/\alpha} \in K(1, 1)$, we can write

$$f(z) = \left(\frac{1 + x\omega(z)}{1 - \omega(z)}\right)^{\alpha} \prec \left(\frac{1 + xz}{1 - z}\right)^{\alpha}$$

where $|x| = 1$. Since for $0 < \alpha < 1$ the function $z \to (1 + xz)^{\alpha}(1 - z)^{-\alpha}$ $(x \neq -1)$ is convex univalent, we deduce

$$|a_n| \leqslant \alpha |1 + x| \leqslant 2\alpha \qquad (n = 1, 2, \ldots).$$

For the case $\beta = 0$ we have

THEOREM 4. *If $f \in \Pi_{\alpha}$ where $\alpha > 0$, then*

$$|a_n| \leqslant \binom{\alpha}{n} \qquad \left(1 \leqslant n \leqslant \left[\frac{\alpha}{2}\right] + 1\right), \tag{19}$$

$$|a_n| \leqslant J(\alpha)/n \qquad (n > [\alpha/2] + 1) \tag{20}$$

where

$$J(\alpha) = \left(\left[\frac{\alpha}{2}\right] + 1\right)\binom{\alpha}{[\alpha/2] + 1}. \tag{21}$$

In particular, $(1 + z)^{\alpha}$ is extremal for the first $[\alpha/2] + 1$ coefficients. Note also that $(1 + z^n)^{\alpha/n} \in \Pi_{\alpha}$, so we cannot do better than α/n for the nth coefficient.

PROOF. Since $\operatorname{Re}(zf'(z)/f(z)) < \frac{1}{2}\alpha$, we can write

$$zf'(z)/f(z) = \alpha\omega(z)/(1 + \omega(z)),$$

where $\omega(0) = 0$, $|\omega(z)| < 1$. We deduce that

$$\left|\sum_{k=0}^{\infty}(k+1)a_{k+1}z^k\right| \leqslant \left|\sum_{k=0}^{\infty}(k-\alpha)a_k z^k\right| \qquad (|z|<1).$$

As shown by Clunie [5] this inequality implies

$$\sum_{k=0}^{n}(k+1)^2|a_{k+1}|^2 \leqslant \sum_{k=0}^{n}(k-\alpha)^2|a_k|^2 \qquad (n=0,1,2,\ldots).$$

Hence

$$(n+1)^2|a_{n+1}|^2 \leqslant \sum_{k=0}^{n}(\alpha^2-2\alpha k)|a_k|^2.$$

Now equality occurs here when $\omega(z) = z, f(z) = (1+z)^\alpha$; hence

$$(n+1)^2\binom{\alpha}{n+1}^2 = \sum_{k=0}^{n}(\alpha^2-2\alpha k)\binom{\alpha}{k}^2.$$

Since $a_0 = 1$, we obtain in the case $n = 0$, $|a_1| \leqslant \alpha$. Suppose $n \leqslant \frac{1}{2}\alpha$ and assume we have shown that $|a_k| \leqslant \binom{\alpha}{k}$ $(1 \leqslant k \leqslant n)$. Then

$$(n+1)^2|a_{n+1}|^2 \leqslant \sum_{k=0}^{n}(\alpha^2-2\alpha k)\binom{\alpha}{k}^2 = (n+1)^2\binom{\alpha}{n+1}^2,$$

so $|a_{n+1}| \leqslant \binom{\alpha}{n+1}$. By induction this holds up to $n = [\alpha/2]$. If $n > [\alpha/2]$, then

$$(n+1)^2|a_{n+1}|^2 \leqslant \sum_{k=0}^{[\alpha/2]}(\alpha^2-2\alpha k)|a_k|^2 \leqslant \sum_{k=0}^{[\alpha/2]}(\alpha^2-2\alpha k)\binom{\alpha}{k}^2 = J^2(\alpha),$$

and we obtain (20).

REMARK 2. For $0 < \alpha \leqslant 2$ this gives the sharp result

$$|a_n| \leqslant \alpha/n \qquad (n=1,2,\ldots)$$

obtained by Clunie [5] and Pommerenke [8] in the context of meromorphic starlike functions. It seems unlikely that the $J(\alpha)$ estimate is sharp when $\alpha > 2$. A tentative conjecture is that

$$|a_n| \leqslant \begin{cases} \binom{\alpha}{n} & (1 \leqslant n \leqslant [\alpha]), \\ \alpha/n & (n > [\alpha]). \end{cases}$$

REMARK 3. Although $(1+z)^\alpha(1-z)^{-\beta}$ is not extremal for the coefficients for every value of α and β, we conjecture that the weaker *Rogosinski dominance* holds:

$$\sum_{k=1}^{n}|a_k|^2 \leqslant \sum_{k=1}^{n}A_k^2 \qquad (n=1,2,\ldots)$$

for $f(z) = 1 + \sum_1^\infty a_n z^n \in K(\alpha,\beta)$, where $A_n = A_n(\alpha,\beta)$ are the coefficients of $(1+z)^\alpha(1-z)^{-\beta}$. This is true for Π_α $(\alpha > 0)$ by subordination: if $f \in \Pi_\alpha$, then $f(z) \prec (1+z)^\alpha$. If this conjecture is true, it implies that for every α and β, the function $(1+z)^\alpha(1-z)^{-\beta}$ is extremal for the pth integral means of $f \in K(\alpha,\beta)$ $(p > 0)$. We prove this result in the next section.

3. Integral means.

THEOREM 5. *If* $f(z) \in K(\alpha, \beta)$, *then for each convex function* Φ *on* $(-\infty, \infty)$, *we have, for* $0 < r < 1$,

$$\int_{-\pi}^{\pi} \Phi\big(\log|f(re^{i\theta})|\big)\, d\theta \leqslant \int_{-\pi}^{\pi} \Phi\big(\log|k_{\alpha,\beta}(re^{i\theta})|\big)\, d\theta \tag{22}$$

where $k_{\alpha,\beta}(z) = (1+z)^{\alpha}(1-z)^{-\beta}$.

We follow a method similar to the argument of Leung [7], who dealt with the close-to-convex case $\alpha = 1$, $\beta = 3$, making use of Baernstein's star function [**2**]. The proof is elementary in that no use is made of Baernstein's fundamental result that u^* is subharmonic when u is. Instead we require four observations concerning the star function.

LEMMA 1. (a) *If* $u(z)$ *is subharmonic in* $|z| < 1$ *and if* $\omega(z)$ *is analytic with* $\omega(0) = 0$, $|\omega(z)| < 1$, *then*:

$$\big(u(\omega(re^{i\theta}))\big)^* \leqslant \big(u(re^{i\theta})\big)^* \qquad (0 < r < 1, 0 \leqslant \theta \leqslant \pi); \tag{23}$$

(b) *if* u *and* $v \in L^1(-\pi, \pi)$, *then*

$$(u+v)^* \leqslant u^* + v^*; \tag{24}$$

(c) *if* u *and* v *are even on* $[-\pi, \pi]$ *and nondecreasing on* $[-\pi, 0]$, *then*

$$u^* + v^* = (u+v)^*; \tag{25}$$

(d) *suppose that* u *and* $v \in L^1(-\pi, \pi)$ *and*

$$\int_{-\pi}^{\pi} u(t)\, dt = \int_{-\pi}^{\pi} v(t)\, dt, \tag{26}$$

$$u^*(\theta) \leqslant v^*(\theta) \qquad (0 \leqslant \theta \leqslant \pi); \tag{27}$$

Then for every convex function Φ *on* $(-\infty, \infty)$,

$$\int_{-\pi}^{\pi} \Phi(u(t))\, dt \leqslant \int_{-\pi}^{\pi} \Phi(v(t))\, dt. \tag{28}$$

Conversely, (28) *implies both* (26) *and* (27).

PROOF. (a) follows on an application of Riesz's subordination inequality [**6**, p. 11]. (b) is trivial. (c) follows from the observation that $w^*(\theta) = \int_{-\theta}^{\theta} w(t)\, dt$ $(0 \leqslant \theta \leqslant \pi)$ for $w(\theta)$ even on $[-\pi, \pi]$ and nondecreasing on $[-\pi, 0]$. To prove (d) we recall that (27) implies (28) for every *nondecreasing* convex Φ on $(-\infty, \infty)$. Now it can be shown (*exercise*) *that every convex function on* $(-\infty, \infty)$ *can be decomposed into the sum of a nondecreasing convex function on* $(-\infty, \infty)$ *with a nonincreasing convex function on* $(-\infty, \infty)$. Therefore we need to show that (28) holds for every nonincreasing convex Φ on $(-\infty, \infty)$. But then $\Phi(-x)$ is nondecreasing convex and so we require

$$(-u)^*(\theta) \leqslant (-v)^*(\theta) \qquad (0 \leqslant \theta \leqslant \pi). \tag{29}$$

Writing $I = [-\pi, \pi]$ we have

$$(-u)^*(\theta) = \sup_{|E|=2\theta}\left(\int_E -u(t)\,dt\right) = \sup_{|E|=2\theta}\left(-\int_{-\pi}^{\pi} u(t)\,dt + \int_{I-E} u(t)\,dt\right)$$
$$= -\int_{-\pi}^{\pi} u(t)\,dt + u^*(\pi-\theta) \leqslant -\int_{-\pi}^{\pi} v(t)\,dt + v^*(\pi-\theta) = (-v)^*(\theta).$$

Conversely, (28) implies (27) **[2]** and (28) implies (26) by taking $\Phi(x) = x$ and $\Phi(x) = -x$.

LEMMA 2. *If $f \in K(\alpha, \beta)$ we can write*

$$f(z) = (1+\omega_1(z))^\alpha/(1-\omega_2(z))^\beta \qquad (|z|<1), \tag{30}$$

where ω_i are analytic, $\omega_i(0) = 0$ and $|\omega_i(z)|<1$ ($|z|<1$, $i = 1, 2$) (i.e. ω_i are Schwarz functions).

PROOF. By Theorem A we can write $f = gH$ where $g \in \Pi_{\alpha-\beta}$ and $H \in K(\lambda, \lambda)$ ($\lambda = \min(\alpha, \beta)$). It is well known that a function $h \in \Pi_{-2}$ is subordinate to $(1+z)^{-2}$ and, as $g = h^{(\beta-\alpha)/2}$ for some such h by Theorem B(c), g is subordinate to $(1+z)^{\alpha-\beta}$. Also $H = P^\lambda$, where $P \in K(1,1)$, and so is subordinate to $(1+xz)(1-z)^{-1}$ for some x ($|x| = 1$). Thus we can write, for suitable Schwarz functions σ_j,

$$f(z) = \left(\frac{1+\sigma_1(z)}{1-\sigma_2(z)}\right)^\lambda (1+\sigma_3(z))^{\alpha-\beta}.$$

It only remains to show that if $\mu > 0$, $\nu > 0$, then for Schwarz functions τ_i, $(1+\tau_1)^\mu(1+\tau_2)^\nu$ is subordinate to $(1+z)^{\mu+\nu}$. Clearly we may assume that $\mu + \nu = 1$. The result follows on taking logs, since $\log(1+z)$ is convex univalent.

LEMMA 3. *Suppose that $F(z) = 1 + A_1 z + \cdots$, $G(z) = 1 + B_1 z + \cdots$ are analytic and nonzero in $|z|<1$, each having real coefficients with $A_1 > 0$, $B_1 > 0$, and suppose further that the two functions $zF'(z)/F(z)$, $zG'(z)/G(z)$ are typically real in $|z|<1$. Then if $f \prec F$, $g \prec G$, we have, for every convex function Φ on $(-\infty, \infty)$,*

$$\int_{-\pi}^{\pi} \Phi(\log|f(re^{i\theta})g(re^{i\theta})|)\,d\theta \leqslant \int_{-\pi}^{\pi} \Phi(\log|F(re^{i\theta})G(re^{i\theta})|)\,d\theta. \tag{31}$$

PROOF. By Lemma 1(d) we must prove

$$(\log|f(re^{i\theta})g(re^{i\theta})|)^* \leqslant (\log|F(re^{i\theta})G(re^{i\theta})|)^*. \tag{32}$$

((26) holds since both integrals are zero.) By Lemma 1(a), (b) the left expression is

$$\leqslant (\log|F(re^{i\theta})|)^* + (\log|G(re^{i\theta})|)^*. \tag{33}$$

Since F has real coefficients, $\log|F(re^{i\theta})|$ is even on $[-\pi, \pi]$. Also

$$\frac{\partial}{\partial\theta}\log|F(re^{i\theta})| = -\operatorname{Im}\frac{re^{i\theta}F'(re^{i\theta})}{F(re^{i\theta})}$$

is nonzero and has constant sign for $\theta \in (-\pi, 0)$. Fixing θ this sign remains constant when r varies (by continuity), and, hence, letting $r \to 0$, the sign is that of $-A_1 \sin\theta$, i.e. it is positive. Thus $\log|F(re^{i\theta})|$ is increasing on $[-\pi, 0]$. Similarly for $\log|G(re^{i\theta})|$. We obtain (32) by applying Lemma 1(c) to (33).

PROOF OF THEOREM 5. The result follows from Lemmas 2 and 3 by putting $F(z) = (1+z)^{\alpha}$, $G(z) = (1-z)^{-\beta}$.

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