PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 2, June 1983

AN INJECTIVE METRIZATION FOR COLLAPSIBLE POLYHEDRA

JIE-HUA MAI AND YUN TANG

ABSTRACT. In this paper we prove that any finite collapsible polyhedron is injectively metrizable.

A metric space Y is *injective* if every mapping which increases no distance from a subspace of any metric space X to Y can be extended, increasing no distance, over X. Isbell [**2**] proved that every 2-dimensional collapsible polyhedron admits injective metrics. In this paper we generalize the result to any finite collapsible polyhedron, which answers a part of the problem put forward by Isbell [**2, 3**].

Let S be a simplicial complex. According to [**4**], S is called *collapsible* if there is a sequence of increasing subcomplexes $S_0, S_1, \ldots, S_n$ such that $S_0 =$ a point, $S = S_n$ and $S_{i+1} = S_i \cup \{\Delta_i, \tau_i\}$, where Δ_i is an r_i-dimensional simplex with an $(r_i - 1)$-dimensional face τ_i such that $S_i \cap \{\Delta_i, \tau_i\} = \varnothing$, $i = 0, 1, \ldots, n-1$. The polyhedron $|S|$ of a (collapsible) complex S is called a (*collapsible*) *polyhedron*.

Let K be a cubical complex. $I = [0,1]$, $I^{n+1} = I^n \times I$. Metrize K as follows: assume that each k-cube of K is a copy of I^k; define the distance between two points $x, y \in |K|$ so that if x and y are in a common cell, for example, in $|I^k|$, then the distance

$$d(x, y) = \max_i |x_i - y_i|,$$

where $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k) \in |I^k|$; otherwise the distance is the length of the shortest path joining them. Obviously, K then is a convex metric space.

DEFINITION 1. Let K be a cubical complex, Y a connected subset of $|K|$. Y is called a *generalized cuboid* of K if for any cell of K, for example, I^k, either the intersection $Y \cap |I^k| = \varnothing$, or there are s_i, t_i, $0 \leqslant s_i \leqslant t_i \leqslant 1$, $i = 1, \ldots, k$, such that

$$Y \cap |I^k| = \{(y_1, \ldots, y_k) \in I^k \mid s_i \leqslant y_i \leqslant t_i, i = 1, \ldots, k\}.$$

For convenience, write GC for generalized cuboid.

DEFINITION 2. Let K be a cubical complex. K is called *collapsible* if there are a sequence of subcomplexes $K_0, K_1, \ldots, K_n$ of K, and nonempty subcomplexes L_i of K_i, $i = 0, 1, \ldots, n$, such that $K_0 =$ one point, $K = K_n$, and $K_{i+1} = K_i \cup L_i \times I$, where

$$L_i \times I = \{c \times \{0\}, c \times I, c \times \{1\} \mid c \in L_i\}, \qquad I = [0,1],$$

$i = 0, 1, \ldots, n-1$. Such K is called *regular* if each $|L_i|$ is a GC of K_i.

Received by the editors March 23, 1982 and, in revised form, April 26, 1982.
1980 *Mathematics Subject Classification*. Primary 54E35; Secondary 57A15.

REMARK 1. Here for every $c \in L_i$, we always identify c and $c \times \{0\}$. In particular $I^n = I^n \times \{0\} \subset I^{n+1}$.

REMARK 2. Isbell [2] gave a different, rather special definition for collapsible cubical complexes in the 2-dimensional case.

LEMMA 1. *Let S be a collapsible simplicial complex. Then S can be subdivided to a regular collapsible cubical complex K such that the polyhedron of any subcomplex of S is exactly the polyhedron of the corresponding subcomplex of K.*

PROOF. Let the subcomplexes of S, $S_0, S_1, \ldots, S_n$, and the simplex Δ_i, τ_i be as above. Set $K_0 = S_0$. Suppose Lemma 1 is true for $n = i$; we want to show it is true for $S_{i+1} = S = S' \cup \{\Delta_i, \tau_i\}$.

Write $\partial\Delta_i = \Delta_i - \operatorname{Int}\Delta_i$, by the hypothesis of induction, the polyhedron $|S_i|$ is subdivided to a regular collapsible cubical complex M, and $\partial\Delta_i - \operatorname{Int}\tau_i$ is a polyhedron of some subcomplex L of M. Obviously, there exists a homeomorphism f of $(\partial\Delta_i - \operatorname{Int}\tau_i) \times I$ onto Δ_i such that $f(x, 0) = x$ for every $x \in \partial\Delta_i - \operatorname{Int}\tau_i$. By f, one can obtain the cubical subdivision $M' = M \cup L \times I$ of $|S'|$.

Consider an arrangement $c_1, c_2, \ldots, c_m$ of all cubes contained in L so that $\dim c_i \leqslant \dim c_{i+1}$, for $1 \leqslant i \leqslant m$. Set

$$M_\alpha = M \cup \left\{c_j \times I, c_j \times \{1\} \,\middle|\, j = 1, 2, \ldots, \alpha\right\}, \qquad \alpha = 0, 1, \ldots, m;$$

then $M = M_0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M \cup L \times I$. Let $Q_\alpha = c_\alpha \times \{0\} \cup \partial c_\alpha \times I$. It is easy to construct a homeomorphism f_α of $|Q_\alpha \times I|$ onto $|c_\alpha \times I|$, $\alpha = 1, 2, \ldots, m$, such that

$$f_\alpha(x, t, 0) = \begin{cases} (x, 0) & \text{if } x \in c_\alpha,\ t = 0, \\ (x, t) & \text{if } x \in \partial c_\alpha,\ t \in I. \end{cases}$$

Let $P_0 = M_0$, $P_\alpha = P_{\alpha-1} \cup Q_\alpha \times I$, $\alpha = 1, \ldots, m$. By construction each P_α is a cubical subdivision of M_α, and $|Q_\alpha|$ is clearly a GC of $P_{\alpha-1}$. So $M \subset P_1 \subset P_2 \subset \cdots \subset P_m$ is a subsequence of regular cubical complexes. Since $|P_m| \approx |M_m| \approx |S_{i+1}|$, P_m is as desired. □

To study injective metrization we give some properties of GC.

LEMMA 2. *Suppose L is a subcomplex of a cubical complex K, $|L|$ is GC in K, and projection*

$$P\colon |K| \cup |L \times I| \to |K|$$

is given by $p(x) = x$ for $x \in |K|$ and $p(y, t) = y$ for $(y, t) \in |L| \times I$. Let $K' = K \cup L \times I$. If X is a GC of K', then

(i) *$p(X)$ is a GC of K;*

(ii) *if $p(X) \cap |L| = \varnothing$, $X = p(X)$;*

(iii) *if $p(X) \cap |L| \neq \varnothing$ and $X \cap |K| = \varnothing$, then there are $s_0, t_0 \in I$, $s_0 \leqslant t_0$, such that $X = p(X) \times [s_0, t_0]$;*

(iv) *if $p(X) \cap |L| \neq \varnothing$ and $X \cap |K| \neq \varnothing$, i.e. $X \cap |L| \neq \varnothing$, then there is $t_0 \in I$ such that $X = (X \cap |K|) \cup ((p(X) \cap |L|) \times [0, t_0])$.*

PROOF. (i) If $|K| \cap X \neq \emptyset$, it is easy to see that $p(X) = |K| \cap X$ is a GC of K. If $|K| \cap X = \emptyset$, $p(X) \subset |L|$ and hence $p(X)$ is a GC of L. Since $|L|$ is a GC of K, so is $p(X)$.

(ii) If $p(X) \cap |L| = \emptyset$, $X \subset |K|$ and hence $X = p(X)$.

(iii) and (iv) follow easily from $X = (X \cap |K|) \cup (X \cap |L \times I|)$ and the following

CLAIM. If $p(X) \cap |L| \neq \emptyset$, then there are $s_0, t_0 \in I$ such that

$$X \cap (L \times I) = (p(X) \cap |L|) \times [s_0, t_0].$$

It suffices to show that if (x, s) and (y, t) in $|L| \times I$ are points in X, then (y, s) is also a point in X. In fact, take a broken line in X

$$[(x_0, s_0), (x_1, s_1), \dots, (x_n, s_n)]$$

such that $(x, s) = (x_0, s_0)$, $(y, t) = (x_n, s_n)$, and $[(x_{i-1}, s_{i-1}), (x_i, s_i)]$ belong to a common cube, $i = 1, \dots, n$. It successively follows from $(x_0, s_0) \in X$ that $(x, s_0), \dots, (x_{n-1}, s_0), (x_n, s_0) = (y, s)$ are in X. □

For $r \geq 0$, nonempty subsets Y of $|K|$ and X of $|K'|$, write

$$B(Y, r) = \{y \in |K| \mid d(y, Y) \leq r\},$$
$$B'(X, r) = \{x \in |K'| \mid d(x, X) \leq r\}.$$

LEMMA 3. *Let K and L be as in Lemma 2. Suppose that for every GC of K, Y, and* $s \geq 0$, $B(Y, s)$ *is a GC of K. Then for every GC of* $K' = K \cup (L \times I)$, X, *and* $r \geq 0$, $B'(X, r)$ *is a GC of* K'.

PROOF. Let X be a GC of K'. The proof conveniently splits into two cases:

Case 1. $X \cap |L| \neq \emptyset$. By (iv) of Lemma 2, there is $t_0 \in I$ such that

$$X = (X \cap |K|) \cup ((p(X) \cap |L|) \times [0, t_0]).$$

Let $B_1 = B'(X \cap |K|, r)$, $B_2 = B'((p(X) \cap |L|) \times [0, t_0], r)$, it is easy to see

$$\begin{aligned} B'(X, r) &= B_1 \cup B_2 \\ &= (B_1 \cap |K|) \cup (B_1 \cap |L \times I|) \cup (B_2 \cap |K|) \cup (B_2 \cap |L \times I|). \end{aligned}$$

It is obvious that

$$B_1 \cap |L \times I| \subset B_2 \cap |L \times I|, \quad B_2 \cap |K| \subset B_1 \cap |K|,$$
$$B_1 \cap |K| = B(X \cap |K|, r),$$

and

$$B_2 \cap |L \times I| = (B(p(X), r) \cap |L|) \times [0, t_1],$$

where $t_1 = \min\{t_0 + r, 1\}$. Then $B'(X, r) = B(X \cap |K|, r) \cup ((B(p(X), r) \cap |L|) \times [0, t_1])$. Since $X \cap |K|$, $p(X)$ and $|L|$ are GC of K, by the hypothesis, $B(X \cap |K|, r)$ and $B(p(X), r) \cap |L|$ are GC of K. So $B'(X, r)$ is a GC of K. The proof of Case 1 is complete.

Case 2. $X \cap |L| = \emptyset$. Let $r_0 = d(X, |L|)$. One has

$$B'(X, r) = B'(B'(X, r_0), r - r_0) \quad \text{whenever } r_0 \leq r.$$

Case 2(a). $X \cap |K| = \varnothing$. By (iii) of Lemma 2, there are $s_0, t_0 \in I$ such that $X = p(X) \times [s_0, t_0]$. If $r_0 \leqslant r$, let $t_2 = \min\{1, t_0 + r_0\}$. Since $B(p(X), r_0)$ is a GC of K, $B'(X, r_0) = (B(p(X), r_0) \cap |L|) \times [0, t_2]$ is a GC of K. Now $B'(X, r_0) \cap |L| \neq \varnothing$, by Case 1, $B'(X, r)$ is a GC of K'. If $r_0 > r$, similarly,

$$B'(X, r) = \big(B(p(X), r) \cap |L|\big) \times [s_0 - r, t_1]$$

is a GC of K'.

Case 2(b). $X \cap |K| \neq \varnothing$, then $X \subset |K|$. If $r_0 \leqslant r$, $B'(X, r_0) = B(X, r_0)$ has nonempty intersection with $|L|$. By Case 1, $B'(X, r)$ is a GC of K'. If $r_0 > r$, $B'(X, r) = B(X, r)$ is a GC of K'. □

Let K be a cubical complex. K is said to have *property* (P) if any collection of GC of K, $\{X_\alpha \mid \alpha \in A\}$, such that every couple of its members intersect, has a common point.

LEMMA 4. *Let K and L be as in Lemma* 2. *If K has the property* (P), *then $K' = K \cup L \times I$ also has the property* (P).

PROOF. Let $\{X_\alpha \mid \alpha \in A\}$ be a collection of GC of K' such that for each α and β in A, $X_\alpha \cap X_\beta \neq \varnothing$. Then $\{p(X_\alpha)\}$ pairwise intersect in $|K|$, and hence $\bigcap_\alpha p(X_\alpha) \neq \varnothing$. We want to show $\bigcap_{\alpha \in A} X_\alpha \neq \varnothing$.

If $X_\alpha \cap |K| \neq \varnothing$ for each $\alpha \in A$, then

$$\Big(\bigcap_\alpha X_\alpha\Big) \cap |K| = \bigcap_\alpha (X_\alpha \cap |K|) = \bigcap_\alpha p(X_\alpha) \neq \varnothing.$$

Hence $\bigcap_\alpha X_\alpha \neq \varnothing$.

If $X_{\alpha_0} \cap |K| = \varnothing$ for some $\alpha_0 \in A$, then $X_{\alpha_0} \subset |L| \times I$, and $p(X_\alpha) \cap |L| \neq \varnothing$ for each $\alpha \in A$. By (iii) and (iv) of Lemma 3, for each $\alpha \in A$, there are s_α, t_α such that $0 \leqslant s_\alpha \leqslant t_\alpha \leqslant 1$ and

$$X_\alpha = (X_\alpha \cap |K|) \cup \big((p(X_\alpha) \cap |L|) \times [s_\alpha, t_\alpha]\big).$$

Set $s = \sup\{s_\alpha \mid \alpha \in A\}$, $t = \inf\{t_\alpha \mid \alpha \in A\}$. One has $s \leqslant t$. In fact, if not, there are $\alpha_1, \alpha_2 \in A$ such that $s_{\alpha_1} > t_{\alpha_2} \geqslant 0$. Then $X_{\alpha_1} \cap |K| = \varnothing$. Obviously $X_{\alpha_1} = (p(X_{\alpha_1}) \cap |L|) \times [s_{\alpha_1}, t_{\alpha_1}]$ does not intersect with $X_{\alpha_2} = (X_{\alpha_2} \cap |K|) \cup ((p(X_{\alpha_2}) \cap |L|) \times [s_{\alpha_2}, t_{\alpha_2}])$. Contradiction. Then $\bigcap_\alpha X_\alpha = (\bigcap_\alpha p(X_\alpha)) \times [s, t] \neq \varnothing$. □

We have to use an important property of injective metric spaces. That is

LEMMA 5. *Let X be a metric space. Then X is injective if and only if X is convex and any collection of solid spheres in pairwise intersection in X has a common point.*

For proof of Lemma 5 see [**1**].

Now we can obtain our main conclusion.

THEOREM. *Let S be a finite collapsible simplicial complex. Then there is a distance function in S such that S becomes an injective metric space.*

PROOF. By Lemma 1, S can be subdivided to a regular collapsible cubical complex K with its natural metric. Let the sequence of subcomplexes of K,

$$K_0 \subset K_1 \subset \cdots \subset K_n = K,$$

and $L_i \subset K_i$, $i = 0, 1, \ldots, n$, be as in Definition 2. Because $|K|$ is convex, using Lemma 5, we need only show that every solid sphere in $|K|$, $B(x, r) = \{y \in |K| \mid d(x, y) \leqslant r\}$ is a GC of K, and that K has the property (P).

The proof will be by induction on n. If $n = 0$, K_0 = one point, it holds obviously. Suppose it holds for $n = j \geqslant 0$. Then the correctness for $K_{j+1} = K = K' \cup L_j \times I$ easily follows from Lemma 3 and Lemma 4. □

The authors are indebted to Professor J. R. Isbell for guidance.

BIBLIOGRAPHY

1. N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405–439; correction, ibid. **7** (1957), 1729.

2. J. R. Isbell, *Six theorems about injective metric spaces*, Comment. Math. Helv. **39** (1964), 65–76.

3. ______, *s admits an injective metric*, Proc. Amer. Math. Soc. **28** (1971), 259–261.

4. J. R. Stallings, *Lectures on polyhedral topology*, Tata Institute of Fundamental Research, Bombay, 1967.

DEPARTMENT OF MATHEMATICS, GUANGXI UNIVERSITY, NANNING, CHINA

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, BUFFALO, NEW YORK 14214