MATRIX SEMIGROUPS

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ABSTRACT. Let S be a semigroup of matrices over a field such that a power of each element lies in a subgroup (i.e., each element has a Drazin inverse within the semigroup). The main theorem of this paper is that there exist ideals $I_0, \dots, I_t$ of S such that $I_0 \subseteq \dots \subseteq I_t = S$, I_0 is completely simple, and each Rees factor semigroup I_k/I_{k-1} , $k = 1, \dots, t$, is either completely 0-simple or else a nilpotent semigroup. The basic technique is to study the Zariski closure of S , which is a linear algebraic semigroup.

Let S be a semigroup. We say that S is *strongly π -regular* ($s \pi r$) if a power of each element of S lies in a subgroup of S . These semigroups were first studied by Azumaya [1], Drazin [4] and Munn [6]. Recently these semigroups have also been called “quasiperiodic” and “group bound.” We let $\mathcal{J}$, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$, $\mathcal{H}$ denote the usual Green’s relations on S (see [3]). It is well known that $\mathcal{J} = \mathcal{D}$ for $s \pi r$ semigroups. A $\mathcal{J}$ -class J of S is *regular* if it contains an idempotent. We let $E(S)$ denote the set of all idempotents of S . If $a, b \in S$, then $a \mid b$ (a divides b) if $b \in S^1 a S^1$.

Throughout this paper Z^+ will denote the set of all positive integers and K will denote an algebraically closed field. If $n \in Z^+$ then $\mathfrak{M}_n(K)$ will denote the algebra of all $n \times n$ matrices over K . If $a \in \mathfrak{M}_n(K)$ then we let $\rho(a)$ denote the rank of a . If $e, a \in \mathfrak{M}_n(K)$, $e^2 = e$, then we let

$$(1) \quad \det_e(a) = \det(eae + 1 - e),$$

where ‘det’ denotes the determinant and 1 is the identity matrix. It is easily seen that

$$(2) \quad \det_e(a) \neq 0 \quad \text{if and only if } eae \in eSe.$$

We consider $\mathfrak{M}_n(K)$ with its Zariski topology (see [9]). A subset of $\mathfrak{M}_n(K)$ is *closed* if it is the zero set of a collection of polynomials in n^2 variables. By the Hilbert Basis Theorem, closed sets satisfy the descending chain condition. Hence, every nonempty family of closed sets has at least one minimal element with respect to inclusion. If $X \subseteq \mathfrak{M}_n(K)$, then we let $\bar{X}$ denote its closure. A closed (multiplicative) subsemigroup of $\mathfrak{M}_n(K)$ will be called an *algebraic semigroup*. Clearly the closure of any subsemigroup of $\mathfrak{M}_n(K)$ is an algebraic semigroup. It has been pointed out by W. E. Clark that every algebraic semigroup is $s \pi r$ (see [7, Corollary 1.4]). If S is any $s \pi r$ subsemigroup of $\mathfrak{M}_n(K)$, then for all $a \in S$, a^n lies in a subgroup of S .

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LEMMA 1. Let S be an algebraic semigroup and I a closed ideal of S such that S/I is nil. Then S/I is nilpotent.

PROOF. Suppose S/I is not nilpotent. Let $\mathcal{Q} = \{T \mid T \text{ is a closed subsemigroup of } S, T^m \not\subseteq I \text{ for all } m \in \mathbb{Z}^+\}$. Then $S \in \mathcal{Q}$. Let T be a minimal element of $\mathcal{Q}$ with respect to inclusion. Then $\overline{T^2} \in \mathcal{Q}$. So $\overline{T^2} = T$. Thus

$$(3) \quad \overline{T^i} = T \quad \text{for all } i \in \mathbb{Z}^+.$$

Let $a \in T$. Suppose $\overline{Ta} = T$. Then $T = \overline{Ta^i}$ for all $i \in \mathbb{Z}^+$. But $a^i \in I$ for some $i \in \mathbb{Z}^+$. So $T \subseteq I$, a contradiction. Hence $\overline{Ta} \neq T$. Thus

$$(4) \quad \overline{Ta} \notin \mathcal{Q} \quad \text{for all } a \in T.$$

Let $\mathfrak{B} = \{M \mid M \text{ is a closed ideal of } T, TM \not\subseteq I\}$. By (3), $T \in \mathfrak{B}$. Let M be a minimal element of $\mathfrak{B}$ with respect to inclusion. If $T^3M \subseteq I$, then by (3), $\overline{TM} = \overline{T^3M} \subseteq \overline{I} = I$, a contradiction. Hence $T^3M \not\subseteq I$. So for some $a \in T$, $TTaM \subseteq I$. Let $M_0 = \overline{TaM}$. Then M_0 is a closed ideal of T , $M_0 \subseteq M$ and $TM_0 \not\subseteq I$. So $M_0 \in \mathfrak{B}$. Hence $M_0 = M$. Thus $\overline{TaM} = M$. Hence $(\overline{Ta})^j M = M$ for all $j \in \mathbb{Z}^+$. By (4) $(\overline{Ta})^k \subseteq I$ for some $k \in \mathbb{Z}^+$. So $TM \subseteq M = (\overline{Ta})^k M \subseteq I$, a contradiction. Hence S/I is nilpotent.

REMARK 2. It is well known that if S is any nil subsemigroup of $\mathfrak{M}_n(K)$, then it is nilpotent (see [5, Proposition 17.9]). If S is nil, then so is $\overline{S}$. Thus Lemma 1 is a generalization of this result.

LEMMA 3. Let S be an $s \pi r$ semigroup. Then:

- (1) If S_1 is an $s \pi r$ subsemigroup of S , if $e \in E(S_1)$ and if H, H_1 are the $\mathcal{H}$ -classes of e in S and S_1 , respectively, then $H \cap S_1 = H_1$.
- (2) Any ideal of S is an $s \pi r$ semigroup.

PROOF. (1) Clearly $H_1 \subseteq H \cap S_1$. Let $a \in S_1 \cap H$. Then $a^m \mathcal{H} f$ in S_1 for some $f \in E(S_1)$. So $f \in aS_1 \cap S_1a$. In S , $e \mathcal{H} a \mathcal{H} a^m \mathcal{H} f$. So $e = f$. Thus $fa = af = a$. It follows that $a \mathcal{H} f = e$ in S_1 . So $a \in H_1$.

(2) Let I be an ideal of S , $a \in I$. Then $a^m \mathcal{H} e$ in S for some $e \in E(S)$. Let H denote the $\mathcal{H}$ -class of e in S . Then $a^m \in H \cap I$ and $H \cap I$ is an ideal of H . So $H \cap I = H$ and $a^m H e$ in I .

LEMMA 4. Let $a, b, e \in \mathfrak{M}_n(K)$, $ab = e = e^2$, $\rho(a) = \rho(e)$. Then $ea = a$.

PROOF. There exist invertible matrices $c, d \in \mathfrak{M}_n(K)$ such that $a = ced$. So $cedb = e$. Hence $edb = c^{-1}e$. So $ec^{-1}e = c^{-1}e$. Thus $ep(c^{-1})e = p(c^{-1})e$ for any polynomial $p(\lambda)$ over K . But c itself is a polynomial in c^{-1} . So $ce = ece$. Hence $ea = eced = ced = a$.

THEOREM 5. Let S be an $s \pi r$ subsemigroup of $\mathfrak{M}_n(K)$. Let I be an ideal of S such that S/I is nil. Then S/I is nilpotent.

PROOF. Suppose the theorem is false. Let $\mathcal{Q} = \{S_1 \mid S_1 \text{ is a closed subsemigroup of } \mathfrak{M}_n(K), \text{ there exists an } s \pi r \text{ subsemigroup } S_2 \text{ of } S_1 \text{ and an ideal } U \text{ of } S_2 \text{ such that}$

S_2/U is nil but not nilpotent}. Then $\mathfrak{N}_n(K) \in \mathcal{Q}$. Let S_1 be a minimal element of $\mathcal{Q}$ with respect to inclusion. There exists an $s\pi r$ subsemigroup T of S_1 and an ideal V of T such that T/V is nil but not nilpotent. So $E(T) \subseteq V$. Thus $W = TE(T)T \subseteq V$. By [7, Theorem 1.7] T has only finitely many regular $\mathcal{J}$ -classes. So there exist $e_1, \dots, e_m \in E(T)$ such that $W = (Te_1T) \cup \dots \cup (Te_mT)$. Since T is $s\pi r$, T/W is nil. Since $W \subseteq V$, it suffices to show that T/W is nilpotent. Clearly $\bar{W}$ is an ideal of $\bar{T}$ and $\bar{T}/\bar{W}$ is nil. By Lemma 1, $\bar{T}/\bar{W}$ is nilpotent. So for some $\alpha \in Z^+$, $T^\alpha \subseteq T \cap \bar{W}$. Now $T \cap \bar{W}$, being an ideal of T , is $s\pi r$ by Lemma 3. Clearly W is an ideal of $T \cap \bar{W}$. If $\bar{W} \notin \mathcal{Q}$, then $(T \cap \bar{W})/W$, being nil, is also nilpotent. Since $T^\alpha \subseteq T \cap \bar{W}$, T/W is nilpotent, a contradiction. Thus $\bar{W} \in \mathcal{Q}$. Since $\bar{W} \subseteq \bar{T} \subseteq S_1$, we see that

$$(5) \quad \bar{W} = \bar{T} = S_1.$$

Let $W_i = Te_iT$, $i = 1, \dots, m$. Then $W = W_1 \cup \dots \cup W_m$ and each W_i is an ideal of T . So each $\bar{W}_i$ is an ideal of $\bar{T}$ and $\bar{W}_1 \cup \dots \cup \bar{W}_m = \bar{W}$. Let $T_i = \bar{W}_i \cap T$, $i = 1, \dots, m$. Then by (5), $T = T_1 \cup \dots \cup T_m$. Clearly each T_i is an ideal of T and $W_i \subseteq T_i \subseteq \bar{W}_i$. By Lemma 3, each T_i is $s\pi r$. Suppose there exists $\beta \in Z^+$ such that $T_i^\beta \subseteq W$, $i = 1, \dots, m$. Then $T^{\beta m} \subseteq W$, a contradiction. So for some $i \in \{1, \dots, m\}$,

$$(6) \quad T_i^p \not\subseteq W \quad \text{for all } p \in Z^+.$$

Let $e = e_i$, $\rho(e) = k$. Clearly $\rho(x) \leq k$ for all $x \in W_i$. Now an $n \times n$ matrix has rank $\leq k$ if and only if all its minors of order $\geq k + 1$ are zero. Thus $\rho(x) \leq k$ for all $x \in \bar{W}_i$. Since $T_i \subseteq \bar{W}_i$, we see that

$$(7) \quad \rho(u) \leq k \quad \text{for all } u \in T_i.$$

Now $Y = \{u \in \bar{T} \mid \det_e(xay) = 0 \text{ for all } x, y \in T_i\}$ is closed and $e \notin Y$. So $Y \neq \bar{T}$. Let $M = \{a \in T_i \mid a \mid e \text{ in } T_i\}$. Suppose $M \neq \emptyset$. Then M is an ideal of T_i and, hence, is an $s\pi r$ semigroup. We claim that $M \subseteq Y$. For suppose $a \in M$, $a \notin Y$. Then $\det_e(xay) \neq 0$ for some $x, y \in T_i$. So by (2), $exaye \mathcal{H} e$ in $\mathfrak{N}_n(K)$. By Lemma 3, $exaye \mathcal{H} e$ in T_i . So $a \mid e$ in T_i , a contradiction. Thus $M \subseteq Y$. So $\bar{M} \subseteq Y$ and $\bar{M} \neq \bar{T}$. Hence $\bar{M} \notin \mathcal{Q}$. So $M/(M \cap W)$, being a nil semigroup, must be nilpotent. Thus there exists $\gamma \in Z^+$ such that

$$(8) \quad M^\gamma \subseteq W \quad \text{or} \quad M = \emptyset.$$

We claim that $T_i \subseteq W \cup M$. So let $a \in T_i$, $a \notin M$. Then $a \mid e$ in T_i . So $xay = e$ for some $x, y \in T_i$. Let $f = ayex \in E(T_i) \subseteq W$. Now $xfay = e$. So $\rho(a) \geq \rho(f) = \rho(e) = k$. Thus by (7), $\rho(a) = k$. So by Lemma 4, $a = fa \in W$. Hence $T_i \subseteq W \cup M$. Since W is an ideal of T , we see by (8) that $T_i^\gamma \subseteq W$, contradicting (6). This proves the theorem.

COROLLARY 6. *Let S be an algebraic semigroup, I an ideal of S . If S/I is nil, then it must be nilpotent.*

COROLLARY 7. *Let S be an algebraic semigroup, I an ideal of S . Then $\bar{I}/I$ is a nilpotent semigroup.*

PROOF. By [8, Theorem 2.3], $\bar{I}/I$ is nil. So by Theorem 5, $\bar{I}/I$ is nilpotent.

COROLLARY 8. *Let S be an $s\pi r$ subsemigroup of $\mathfrak{M}_n(K)$. Let I be an ideal of S . Then there exists $p \in \mathbb{Z}^+$, $e_1, \dots, e_m \in E(I)$ such that $I^p = I^{p+1} = Se_1S \cup \dots \cup Se_mS$.*

PROOF. Let $M = SE(I)S$. Then, clearly $M \subseteq I^k$ for all $k \in \mathbb{Z}^+$. Since I is $s\pi r$, I/M is nil. By Theorem 5, there exists $p \in \mathbb{Z}^+$ such that $I^p \subseteq M$. So $I^{p+1} \subseteq I^p \subseteq M \subseteq I^{p+1}$. By [7, Theorem 1.7], S has only finitely many regular $\mathcal{J}$ -classes. Hence there exist $e_1, \dots, e_m \in E(I)$ such that $M = Se_1S \cup \dots \cup Se_mS$.

Clark [2] showed that if S is an $s\pi r$ subsemigroup of $\mathfrak{M}_n(K)$, then it has a completely simple kernel consisting of elements of S of minimal rank. The author [7, Theorem 1.10] has shown that S has ideals $I_1, \dots, I_t$ such that $I_1 \subseteq \dots \subseteq I_t = S$, I_1 is completely simple, and each Rees factor semigroup I_k/I_{k-1} is either completely 0-simple or nil. By Theorem 5, the nil factors are all nilpotent.

THEOREM 9. *Let S be an $s\pi r$ subsemigroup of $\mathfrak{M}_n(K)$. Then:*

(i) *There exists $m \in \mathbb{Z}^+$, $m \geq 2$, ideals $I_1, \dots, I_t$ of S such that $I_1 \subseteq \dots \subseteq I_t = S$. I_1 is the completely simple kernel of S , and for $k = 2, \dots, t$, either $I_k^m \subseteq I_{k-1}$ or I_k/I_{k-1} is a completely 0-simple semigroup.*

(ii) *S does not have a chain of more than m' principal right or left ideals.*

(iii) *S does not have a chain of more than m^{2t} principal two-sided ideals.*

(iv) *S satisfies the ascending and descending chain conditions on principal right, left and two-sided ideals.*

PROOF. We have already proven (i) above. So we next prove (ii). Let $q = m'$. Suppose there exist $a_1, \dots, a_{q+1} \in S$ such that

$$(9) \quad a_1S^1 \not\supseteq a_2S^1 \not\supseteq \dots \not\supseteq a_{q+1}S^1.$$

Then there exist $x_1, \dots, x_q \in S$ such that $a_jx_j = a_{j+1}$, $j = 1, \dots, q$. We first claim that for some $1 \leq \alpha < \beta < \gamma \leq q$, $x_\alpha \cdots x_\beta \mathcal{R} x_\alpha \cdots x_\gamma$. We prove this by induction on t . Let $u_1 = x_1 \cdots x_m$, $u_2 = x_{m+1} \cdots x_{2m}, \dots$. If each $u_p \in I_{t-1}$, $p = 1, \dots, m^{t-1}$, then we are done by induction. Suppose then that some $u_p \notin I_{t-1}$. If S/I_{t-1} is nilpotent, then $u_p \in S^m \subseteq I_{t-1}$, a contradiction. So S/I_{t-1} is completely 0-simple. Clearly $x_{pm}, x_{pm-1}, x_{pm-1}x_{pm} \in S \setminus I_{t-1}$. Since S/I_{t-1} is completely 0-simple, it follows that $x_{pm-1}x_{pm} \mathcal{R} x_{pm-1}$. Thus our claim is true. So for some $1 \leq \alpha < \beta < \gamma \leq q$, $x_\alpha \cdots x_\beta \mathcal{R} x_\alpha \cdots x_\gamma$. Thus

$$a_{\beta+1} = a_\alpha x_\alpha \cdots x_\beta \mathcal{R} a_\alpha x_\alpha \cdots x_\gamma = a_{\gamma+1}.$$

Hence $a_{\beta+1}S^1 = a_{\gamma+1}S^1$, contradicting (9). This proves (ii).

Now we prove (iii). Let $q = m'$, $\theta = q^2 + 1$. Suppose there exist $a_1, \dots, a_\theta \in S$ such that

$$(10) \quad S^1a_1S^1 \supseteq \dots \supseteq S^1a_\theta S^1.$$

There exist $x_1, \dots, x_{\theta-1}, y_1, \dots, y_{\theta-1} \in S^1$ such that $x_ja_jy_j = a_{j+1}$, $j = 1, \dots, \theta - 1$. Let $x_0 = y_0 = 1$. Let $b_{j+1} = a_1y_0 \cdots y_j$, $c_{j+1} = x_j \cdots x_0a_1$, $j = 0, \dots, \theta - 1$. So

$$b_1S^1 \supseteq b_2S^1 \supseteq \dots \supseteq b_\theta S^1, \quad S^1c_1 \supseteq S^1c_2 \supseteq \dots \supseteq S^1c_\theta.$$

Let $X = \{b_1 S^1, \dots, b_\theta S^1\}$, $Y = \{S^1 c_1, \dots, S^1 c_\theta\}$. By (ii) $|X| \leq q$, $|Y| \leq q$. Let $W = \{1, \dots, \theta\}$. Define $\phi: W \rightarrow X \times Y$ as $\phi(j) = (b_j S^1, S^1 c_j)$. Since $\theta = q^2 + 1$, there exist $\alpha, \beta \in W$ such that $\alpha \neq \beta$, $\phi(\alpha) = \phi(\beta)$. So $b_\alpha S^1 = b_\beta S^1$, $S^1 c_\alpha = S^1 c_\beta$. Thus

$$\begin{aligned} S^1 a_\alpha S^1 &= S^1 x_\alpha \cdots x_0 a_1 y_0 \cdots y_\alpha S^1 = S^1 c_\alpha y_0 \cdots y_\alpha S^1 = S^1 c_\beta y_0 \cdots y_\alpha S^1 \\ &= S^1 x_\beta \cdots x_0 a_1 y_0 \cdots y_\alpha S^1 = S^1 x_\beta \cdots x_0 b_\alpha S^1 = S^1 x_\beta \cdots x_0 b_\beta S^1 \\ &= S^1 x_\beta \cdots x_0 a_1 y_0 \cdots y_\beta S^1 = S^1 a_\beta S^1. \end{aligned}$$

Thus $S^1 a_\alpha S^1 = S^1 a_\beta S^1$, contradicting (10).

Clearly (iv) is a consequence of (ii) and (iii). This proves the theorem.

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