FINITE DIMENSIONAL REPRESENTATIONS OF JB ALGEBRAS

HORST BEHNCKE

ABSTRACT. Let Λ be the set of isomorphism classes of finite dimensional JB factors, ordered by inclusion. For any $\alpha \in \Lambda$ and JB algebra $\mathcal{Q}$ denote by ${}_{\alpha}\hat{\mathcal{Q}}$ the set of equivalence classes of factor representations of $\mathcal{Q}$ into a JB factor of type α . Then ${}_{\alpha}\hat{\mathcal{Q}}$ is closed.

The aim of this note is to extend the results on finite dimensional representations of C^* -algebras [4, 3.6] to the JB situation. The finite dimensional simple JB algebras have all been classified by Jordan, von Neumann and Wigner [6]. They are

J_1 : The real numbers $\mathbf{R}$.

$J_{2,n}$: The spin factor over an n -dimensional real Hilbert space for $n \geq 2$.

$J_{3,k}$: The hermitian 3×3 matrices over $\mathbf{R}$ ($k = 1$), $\mathbf{C}$ the complex numbers ($k = 2$), $\mathbf{H}$ the quaternions ($k = 4$) or $\mathbf{O}$ the Cayley numbers ($k = 8$).

$J_{n,k}$: The hermitian $n \times n$ matrices, $n \geq 4$, over $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$.

Let Λ be this classifying set of finite dimensional simple JB algebras. For $\alpha, \beta \in \Lambda$ write $\alpha \leq \beta$ if J_{α} can be considered a subalgebra of J_{β} .

In this note JB algebras will not necessarily be unital and they are defined as in [3]. It is, however, possible to adjoin a unit [3]. To every pure state ρ on a JB algebra $\mathcal{Q}$ one associates canonically a dense representation π_{ρ} of $\mathcal{Q}$ into a type I JW factor [1, Chapter 5]. Therefore we define the primitive ideal space of $\mathcal{Q}$ by $\text{Prim } \mathcal{Q} = \{\ker \pi \mid \pi \text{ is a dense representation of } \mathcal{Q} \text{ into a type I JBW factor}\}$. With the Jacobson topology $\text{Prim } \mathcal{Q}$ is a T_0 space [4, 3.1.3]. It is even a T_1 space if all dense type I factor representations are finite dimensional.

As in [4, Chapter 3.6] let $\hat{\mathcal{Q}}_{\alpha}$, respectively ${}_{\alpha}\hat{\mathcal{Q}}$, $\alpha \in \Lambda$, denote the set of all equivalence classes of dense factor representations onto J_{α} , respectively onto J_{β} , with $\beta \leq \alpha$. In this case one can identify $\hat{\mathcal{Q}}_{\alpha}$ with a part of $\text{Prim } \mathcal{Q}$. This will be done henceforth.

THEOREM. ${}_{\alpha}\hat{\mathcal{Q}}$ is closed for all $\alpha \in \Lambda$.

PROOF. (a) We may assume that $\mathcal{Q}$ is unital and that $\mathcal{Q}$ has a dense set of dense representations of type β with $\beta \leq \alpha$. In C^* -algebra theory the closedness of ${}_{\alpha}\hat{\mathcal{Q}}$ is usually achieved by the Amitsur Levitzki Theorem [4, Chapter 3.6]. It states in

Received by the editors June 22, 1982 and, in revised form, September 20, 1982.

1980 *Mathematics Subject Classification*. Primary 46L05.

Key words and phrases. JB algebras, finite dimensional representations.

©1983 American Mathematical Society
0002-9939/82/0000-1077/\$01.50

particular: Let M_n be the algebra of n by n matrices with complex entries and for $x_1, \dots, x_r \in M_n$ let

$$P_r = \sum_{\sigma \in \mathfrak{S}_r} (\text{sign } \sigma) x_{\sigma(1)} \cdots x_{\sigma(r)}.$$

Then $P_r = 0$ for all choices of $x_1, \dots, x_r \in M$ iff $r \geq 2n$.

This result however is not available for Jordan algebras [7]. B. D. Smith has modified the concept of polynomial identities to the Jordan algebra situation and has shown in [7] a weak version of the Amitsur Levitzki result.

Let π be a dense factor representation of $\mathcal{Q}$ into M . Then M satisfies the same standard Jordan polynomial identity as $\mathcal{Q}$, respectively J_α . Because of the halving lemma [1, Theorem 6.9 and 7, Proposition 5] we see that M is a finite type I JBW factor. Thus $\mathcal{Q}$ is a type I finite JB algebra.

(b) Let $\alpha = (n, k)$; then Lemma 6 of [2] shows that any dense representation π of $\mathcal{Q}$ is even of type $I_{\leq n}$. $\pi(\mathcal{Q})$ is finite dimensional even, because the infinite dimensional spin factor can be excluded by means of polynomial identities as in [2]. Thus $\pi(\mathcal{Q}) = J_\beta$. Let $x_1, \dots, x_s \in \mathcal{Q}$ such that $\pi(x_1), \dots, \pi(x_s)$ generate J_β . By restricting to the subalgebra of $\mathcal{Q}$, which is generated by $x_1, \dots, x_s$, if necessary, we may even assume $\mathcal{Q}$ to be separable. If $\beta \leq \alpha$ we are finished. Thus let $\beta \not\leq \alpha$ and let $\pi_i \in \hat{\mathcal{Q}}$ with $\pi_i \rightarrow \pi$. We may assume $\{\pi_i\}$ to be universal with $\bigcap \ker \pi_i = (0)$. This net has only finitely many cluster points [5, Lemma 2.4] $\rho_1 = \pi, \rho_2, \dots, \rho_k$. There exists an element $x \in \mathcal{Q}$ with $\pi(x) \neq 0$ and $\rho_i(x) = 0, i = 2, \dots, k$; because $\hat{\mathcal{Q}}$ is a T_1 space even. With this x one constructs, by averaging of x^2 , an element $e \in \mathcal{Q}$ with $\pi(e)$ the unit of $\pi(\mathcal{Q})$ and $\rho_i(e) = 0$ for $i = 2, \dots, k$. Replacing $\mathcal{Q}$ by $U_e(\mathcal{Q})$ if necessary, we may thus assume that π is the only cluster point of the π_i . The proof can now be completed as in [5, Lemma 2.5] by using the functional calculus of continuous functions and the trace. Actually one shows that $\pi(\mathcal{Q})$ can be embedded approximately in $\pi_i(\mathcal{Q})$ for large i .

COROLLARY 1. *Let $\mathcal{Q}$ be a JB algebra such that any dense type I factor representation is of type I_n . Then $\text{Prim } \mathcal{Q} = \hat{\mathcal{Q}}$ is locally compact.*

PROOF. Since for each $x \geq 0, \mathcal{Q} \ni \pi \rightarrow \text{Tr } \pi(x)$ and $\text{Tr } \pi(1 - x) = n - \text{Tr } \pi(x)$ are lower semicontinuous [2, Lemma 6], $\hat{\mathcal{Q}} \ni \pi \rightarrow \text{Tr } \pi(x)$ is even continuous. Thus $\hat{\mathcal{Q}}$ is a T_2 space [4, 4.53].

Algebras of this type will be called n -homogeneous. The term α -homogeneous, $\alpha \in \Lambda$, will be used in a similar fashion.

Let $\mathcal{Q}$ be an n -homogeneous JB algebra and let $P_0 \in \hat{\mathcal{Q}}$ determine a representation of type (n, k) . In $J_{(n,k)}$ one can define a canonical system of matrix elements by

$$(i) S_{ii}^1 = e_{ii}, i = 1, \dots, n,$$

$$(ii) s_{ij}^l = b_l e_{ij} + \bar{b}_l e_{ij}, i \neq j, i, j = 1, \dots, n, l = 1, \dots, k, b_l = 1,$$

where the e_{ij} are the usual matrix units and where $\{b_l | l = 1, \dots, k\}$ is a canonical basis of $\mathbf{R}, \mathbf{C}, \mathbf{H}$ or $\mathbf{O}$. Arguing as in [2] one can determine an open neighbourhood U of P_0 and elements $s_{ij}^l, i, j = 1, \dots, n$ and $l = 1, \dots, k$, such that these elements satisfy the same relations at each $P \in U$ as the canonical system of matrix units.

This shows without difficulty that Theorem 1 of [2] carries over with only minor modification to this situation. For short we write this as

COROLLARY 2. *α -homogeneous JB algebras correspond uniquely up to isomorphism to fibre bundles $\{\mathfrak{B}, pr, \hat{\mathcal{A}}, J_\alpha, \text{Aut } J_\alpha\}$.*

The above results also show that locally, n -homogeneous algebras with $n \geq 4$, look like subalgebras of $(n, 4)$ homogeneous algebras. Globally this is probably not true. For $n = 2, 3$ a similar result holds.

REFERENCES

1. E. M. Alfsen, F. W. Schultz and E. Størmer, *A Gelfand-Naimark Theorem for Jordan algebras*, Adv. in Math. **28** (1978), 11–56.
2. H. Behncke and W. Bös, *JB algebras with an exceptional ideal*, Math. Scand. **42** (1978), 306–312.
3. H. Behncke, *Hermitian Jordan Banach algebras*, J. London Math. Soc. (2) **20** (1979), 327–333.
4. J. Dixmier, *Les C^* -algèbres et leur représentations*, Gauthier-Villars, Paris, 1964.
5. J. M. G. Fell, *The dual spaces of C^* -algebras*, Trans. Amer. Math. Soc. **94** (1960), 365–403.
6. P. Jordan, J. von Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. of Math. (2) **35** (1934), 29–64.
7. B. D. Smith, *A standard Jordan polynomial*, Comm. Algebra **5** (1977), 207–218.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSNABRUCK, OSNABRUCK, FEDERAL REPUBLIC OF GERMANY