SHIFT BASIC SEQUENCES IN THE WIENER DISC ALGEBRA

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ABSTRACT. Let $W(D)$ denote the set of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $\sum_{n=0}^{\infty} |a_n| < +\infty$. It is shown that for any positive integer k the k -shifted sequence $\{z^{kn} \cdot f(z)\}_{n=0}^{\infty}$ is a basic sequence in $W(D)$ equivalent to the basis $\{z^n\}_{n=0}^{\infty}$ if and only if $f(z)$ has no set of k symmetrically distributed zeros on the circle $|z| = 1$.

1. Introduction. A sequence $\{x_n\}_{n=0}^{\infty}$ in a Banach space X is said to be a *basis* for X if every $x \in X$ can be written uniquely as a convergent series $x = \sum_{n=0}^{\infty} a_n x_n$. The sequence $\{x_n\}_{n=0}^{\infty}$ is called a *basic sequence* [4, p. 27] if it forms a basis for its closed linear span in X . If $\{x_n\}_{n=0}^{\infty}$ is a basis for X and $\{y_n\}_{n=0}^{\infty}$ a basis for Y , we say $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are *equivalent* if the operator $T: X \rightarrow Y$ defined by $Tx_n = y_n$ is a linear homeomorphism, i.e. if $\sum_{n=0}^{\infty} a_n x_n$ converges in X if and only if $\sum_{n=0}^{\infty} a_n y_n$ converges in Y [4, p. 70].

Let $W(D)$ denote the Wiener disc algebra of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $\sum_{n=0}^{\infty} |a_n| < +\infty$. Defining the norm on $W(D)$ by $\|f\| = \sum_{n=0}^{\infty} |a_n|$ and multiplication as that of analytic functions, $W(D)$ is a Banach algebra with identity 1 which is isometrically isomorphic to the (complex) sequence space l^1 under the mapping $\sum_{n=0}^{\infty} a_n z^n \leftrightarrow \{a_n\}_{n=0}^{\infty}$. It follows that the sequence $\{z^n\}_{n=0}^{\infty}$ is a basis for $W(D)$ which is equivalent to the unit vector basis $\{e_n\}_{n=0}^{\infty}$ for l^1 and which is referred to as the *standard basis* for $W(D)$.

Now this basis $\{z^n\}_{n=0}^{\infty}$ is simply the set of successive shifts $\{1, z \cdot 1, \dots, z^n \cdot 1, \dots\}$ of the function $1 \in W(D)$. Thus a very natural and interesting problem which arises is that of determining the basis behavior of more general shift sequences in $W(D)$, and particularly that of distinguishing those equivalent to the standard basis $\{z^n\}_{n=0}^{\infty}$. In its greatest generality this latter problem can be stated as follows:

Given a function $f(z)$ in $W(D)$ and a positive integer k , when is the " k -shifted" sequence $\{z^{kn} \cdot f(z)\}_{n=0}^{\infty}$ a basic sequence in $W(D)$ equivalent to the standard basis $\{z^n\}_{n=0}^{\infty}$?

The purpose of this paper is to provide a complete solution to the problem. In §2 it is shown that the sequence $\{z^{kn} \cdot f(z)\}_{n=0}^{\infty}$ is basic in $W(D)$ and equivalent to $\{z^n\}_{n=0}^{\infty}$ if and only if $f(z)$ has no set of k zeros on the circle $|z| = 1$ which are "symmetrically distributed", i.e. if $\sum_{i=1}^k |f(\omega^i z)| \neq 0$ for all z , where ω is a primitive k th root of 1. The solution to the general problem of characterizing those functions $f(z)$ for which $\{z^{kn} \cdot f(z)\}_{n=0}^{\infty}$ is *some* basic sequence in $W(D)$ (not necessarily

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equivalent to $\{z^n\}_{n=0}^\infty$) is considerably more difficult even when $k = 1$, and is known at this time only in special cases, e.g., when $f(z)$ is a polynomial. We briefly discuss the problems and possibilities inherent in this general case in §3.

Finally, we note that problems of this type concerning the description of 1-shifted sequences which form bases for function spaces other than $W(D)$ have been extensively studied (see, e.g., [1] and its bibliography). In the space $W(D)$ itself it was shown earlier [2] that all shift bases for $W(D)$ are equivalent to $\{z^n\}_{n=0}^\infty$. Hence the present paper forms the natural completion to this aspect of the study of shift sequences in $W(D)$.

2. Main results. In this section we characterize those functions $f(z)$ for which a shift sequence of the form $\{z^{kn} \cdot f(z)\}_{n=0}^\infty$ in $W(D)$ is a basic sequence equivalent to $\{z^n\}_{n=0}^\infty$. In particular we give such a characterization in terms of the location of zeros of $f(z)$ on the unit circle $|z| = 1$, thereby linking the function-theoretic properties of $f(z)$ to the basic structure of $W(D)$.

THEOREM. *Let $f(z)$ be a nonzero function in $W(D)$ and k a positive integer. Then the sequence $\{z^{kn} \cdot f(z)\}_{n=0}^\infty$ is a basic sequence in $W(D)$ equivalent to $\{z^n\}_{n=0}^\infty$ if and only if $\sum_{i=1}^k |f(\omega^i z)| \neq 0$ for all $|z| = 1$, where ω is a primitive k th root of unity.*

I.e. $\{z^{kn} \cdot f(z)\}_{n=0}^\infty$ is equivalent to $\{z^n\}_{n=0}^\infty$ if and only if $f(z)$ does not have k zeros which are "symmetrically distributed" around the circle $|z| = 1$.

PROOF. In the interest of simplicity of notation we prove the theorem for the case $k = 2$, then indicate the straightforward adaptation of the argument to all other values of k . Thus we will characterize the equivalence of a sequence of the form $\{z^{2n} \cdot f(z)\}_{n=0}^\infty$ to $\{z^n\}_{n=0}^\infty$.

Since $f(z) \not\equiv 0$ we may suppose $f(z)$ has the form $f(z) = 1 + a_1 z + a_2 z^2 + \dots$. If $\{z^n\}_{n=0}^\infty$ denotes the usual basis for $W(D)$ then the linear operator $T: W(D) \rightarrow W(D)$ defined by $T(z^n) = z^{2n} \cdot f(z)$, $n = 0, 1, 2, \dots$, is continuous since $\sup_n \|T(z^n)\| = \|f(z)\| < +\infty$. It follows that the sequence $\{z^{2n} \cdot f(z)\}_{n=0}^\infty$ is a basic sequence in $W(D)$ equivalent to $\{z^n\}_{n=0}^\infty$ if and only if there is an $\epsilon > 0$ for which $\|T(g(z))\| \geq \epsilon$ for all $g(z)$ in $W(D)$ with $\|g(z)\| = 1$.

If we define $f_1(z) = \sum_{n=0}^\infty a_{2n} z^n$ and $f_2(z) = \sum_{n=0}^\infty a_{2n+1} z^n$, then clearly for any $g(z)$ in $W(D)$ we have

$$T(g(z)) = g(z^2) \cdot f(z) = g(z^2) f_1(z^2) + z \cdot g(z^2) f_2(z^2).$$

It follows that

$$\|Tg(z)\| = \|g(z^2) \cdot f_1(z^2)\| + \|zg(z^2) f_2(z^2)\| = \|g(z) \cdot f_1(z)\| + \|g(z) \cdot f_2(z)\|,$$

and by our comments above we have: *The sequence $\{z^{2n} \cdot f(z)\}_{n=0}^\infty$ is a basic sequence in $W(D)$ equivalent to $\{z^n\}_{n=0}^\infty$ if and only if there is an $\epsilon > 0$ such that $\|g(z) \cdot f_1(z)\| + \|g(z) \cdot f_2(z)\| \geq \epsilon$ for all $g(z)$ in $W(D)$ with $\|g(z)\| = 1$.*

Thus we need only show that the assumptions of the theorem imply this last holds.

Suppose, then, that the condition of the theorem is satisfied for $k = 2$, i.e. $|f(z)| + |f(-z)| \neq 0$ for every $|z| = 1$. It follows that $|f_1(z)| + |f_2(z)| \neq 0$ for every

$|z| = 1$, since if $f_1(z) = f_2(z) = 0$, let $w = z^{1/2}$ so that $|w| = 1$ and

$$\begin{aligned} f(w) &= 1 + a_1w + a_2w^2 + \cdots = (1 + a_2z + a_4z^2 + \cdots) + w(a_1 + a_3z + \cdots) \\ &= f_1(z) + wf_2(z) = 0, \end{aligned}$$

while

$$\begin{aligned} f(-w) &= 1 - a_1w + a_1w^2 - \cdots = (1 + a_1z + a_2z^2 + \cdots) - w(a_1 + a_3z + \cdots) \\ &= f_1(z) - wf_2(z) = 0, \end{aligned}$$

a contradiction to our assumption.

Now let A denote the Banach algebra of all functions $f(\theta)$ on $[0, 2\pi]$ whose Fourier series is of the form $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{-in\theta}$, where $\sum_{n=-\infty}^{\infty} |a_n| < +\infty$, and for which $\|f(\theta)\| = \sum_{n=-\infty}^{\infty} |a_n|$ (see, e.g., [3, p. 195]). Clearly we may identify $W(D)$ with the subspace of A consisting of those functions $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{-in\theta}$ for which $a_n = 0$ for $n < 0$.

Regarding f_1 and f_2 as functions in A , by the above we have that f_1 and f_2 have no common zero on $[0, 2\pi]$. Since the proper maximal ideals in A are sets of the form $\{h \in A \mid h(t_0) = 0\}$ for some $t_0 \in [0, 2\pi]$ [3, p. 195], it follows that f_1 and f_2 are not both contained in any proper maximal ideal of A and, hence, that the ideal in A generated by f_1 and f_2 must be A itself. In particular, there exist functions p and q in A for which $p \cdot f_1 + q \cdot f_2 = 1$.

Now suppose it is *not* true that there exists an $\epsilon > 0$ such that $\|g(z)\| = 1$ implies $\|f_1(z) \cdot g(z)\| + \|f_2(z) \cdot g(z)\| \geq \epsilon$. Then there exists a sequence $\{g_n(z)\}$ in $W(D)$ for which $\|g_n(z)\| = 1$ for all n , but $\|g_n(z) \cdot f_1(z)\| \rightarrow 0$ and $\|g_n(z) \cdot f_2(z)\| \rightarrow 0$. Regarding $\{g_n\}_{n=1}^{\infty}$ as a set of functions in A (as above) and noting that multiplication is continuous on A , we then have

$$g_n = g_n \cdot 1 = g_n(p f_1 + q f_2) = p(g_n f_1) + q(g_n f_2),$$

where this last goes to zero as $n \rightarrow \infty$ since $\|g_n f_1\| \rightarrow 0$ and $\|g_n f_2\| \rightarrow 0$. But this contradicts the fact that $\|g_n\| = 1$ for all n ; hence it must be that for some $\epsilon > 0$, $\|f_1(z)g(z)\| + \|f_2(z)g(z)\| \geq \epsilon$ whenever $\|g(z)\| = 1$, and the sequence $\{z^{2n} \cdot f(z)\}_{n=0}^{\infty}$ is basic and equivalent to $\{z^n\}$.

Conversely, suppose $|f(z_0) + f(-z_0)| = 0$ for some $|z_0| = 1$. Then for every $n = 1, 2, \dots$ let $w_n(z)$ and $v_n(z)$ be polynomials for which $\|f_1(z) - w_n(z)\| < 1/4n$, $\|f_2(z) - v_n(z)\| < 1/4n$, and $w_n(z_0) = v_n(z_0) = 0$, so that $w_n(z) = (z - z_0)q_n(z)$ and $v_n(z) = (z - z_0)t_n(z)$ for some polynomials $q_n(z)$ and $t_n(z)$. If r is any positive integer then

$$(z - z_0) \cdot \frac{1}{(r+1)} \left(1 + \frac{1}{z_0}z + \frac{1}{z_0^2}z^2 + \cdots + \frac{1}{z_0^r}z^r \right) = \frac{1}{(r+1)} \left(\frac{1}{z_0^r}z^{r+1} - z_0 \right);$$

hence, by the above,

$$\begin{aligned} \left\| w_n(z) \cdot \frac{1}{(r+1)} \left(1 + \frac{1}{z_0}z + \cdots + \frac{1}{z_0^r}z^r \right) \right\| \\ = \left\| q_n(z) \cdot \frac{1}{(r+1)} \left(\frac{1}{z_0^r}z^{r+1} - z_0 \right) \right\| \leq \frac{2}{(r+1)} \|q_n(z)\|. \end{aligned}$$

For every n let r_n be an integer for which $2(r_{n+1})^{-1}\|q_n(z)\| < 1/4n$. Then whenever $r \geq r_n$,

$$\left\| w_n(z) \cdot \frac{1}{(r+1)} \left(1 + \frac{1}{z_0}z + \cdots + \frac{1}{z_0^r}z^r \right) \right\| < \frac{1}{4n}.$$

Now for every integer $r \geq 1$ define the polynomial

$$p_r(z) = \frac{1}{(r+1)} \left(1 + \frac{1}{z_0}z + \cdots + \frac{1}{z_0^r}z^r \right),$$

so that $\|p_r(z)\| = 1$ for all r and, hence, for which

$$\|(f_1(z) - w_n(z))p_r(z)\| \leq \|f_1(z) - w_n(z)\| \cdot \|p_r(z)\| < 1/4n$$

(by definition of $w_n(z)$). It follows that

$$\|f_1(z)p_r(z)\| < \frac{1}{4n} + \|w_n(z)p_r(z)\| < \frac{1}{4n} + \frac{1}{4n} = \frac{1}{2n}.$$

An identical argument applied to the polynomials $v_n(z) = (z - z_0)t_n(z)$ shows that for every n there is an integer s_n such that if r is any integer $\geq s_n$, then one also has $\|f_2(z)p_r(z)\| < 1/2n$. If for every $n = 1, 2, \dots$ we set $i_n = \max\{r_n, s_n\}$, then it follows that $\|f_1(z)p_{i_n}(z)\| < 1/2n$ and $\|f_2(z)p_{i_n}(z)\| < 1/2n$ for all n . Hence the sequence $\{x_n(z)\}_{n=1}^\infty$ in $W(D)$, defined by

$$x_n(z) = \frac{1}{i_n + 1} \left(1 + \frac{1}{z_0}z + \cdots + \frac{1}{z_0^{i_n}}z^{i_n} \right),$$

satisfies $\|x_n(z)\| = 1$ and

$$\|Tx_n\| = \|f_1(z)p_{i_n}(z)\| + \|f_2(z)p_{i_n}(z)\| < 1/2n + 1/2n = 1/n$$

for all n . That is, T is not bounded below on $W(D)$, and the sequence $\{z^{2n} \cdot f(z)\}_{n=0}^\infty$ is not equivalent to $\{z^n\}_{n=0}^\infty$ in $W(D)$.

The proof of the theorem for $k > 2$ is entirely analogous. The condition that $\sum_{i=1}^k |f(\omega^i(z))| \neq 0$ for all $|z| = 1$ can easily be shown to be equivalent to the statement that the k functions

$$\begin{aligned} f_1(z) &= 1 + a_k z + a_{2k} z^2 + \cdots, \\ f_2(z) &= a_1 + a_{k+1} z + a_{2k+1} z^2 + \cdots, \\ f_k(z) &= a_{k-1} + a_{2k-1} z + \cdots \end{aligned}$$

have no common zero on the circle $|z| = 1$. Then using the obvious “ k -fold” analogue of the argument given above for the case $k = 2$, this last condition is shown to be equivalent to the assertion that the operator $T: W(D) \rightarrow W(D)$, defined by $Tz^n = z^{kn} \cdot f(z)$ for all n , is bounded below on $W(D)$, or, what is the same, that $\{z^{kn} \cdot f(z)\}_{n=0}^\infty$ is a basic sequence in $W(D)$ equivalent to $\{z^n\}_{n=0}^\infty$.

Finally, the case $k = 1$ can be treated even more simply. Here the condition $f(z) \neq 0$ for all $|z| = 1$ is equivalent (by the Wiener-Levy Theorem [3, p. 196]) to the statement that $f^{-1}(z) \in A$, and this latter is easily shown to be equivalent to the fact that $T: W(D) \rightarrow W(D)$, defined by $T(z^n) = z^n f(z)$, is bounded below on l^1 .

3. Remarks. The theorem given in §2 provides a complete function-theoretic description of the shift basic sequences in $W(D)$ which are equivalent to $\{z^n\}_{n=0}^\infty$. In view of this result it is natural to conjecture that *arbitrary* shift basic sequences in $W(D)$ are also characterized by properties of the zeros of their generating functions on the circle $|z| = 1$. Further evidence to support this conjecture is provided by our earlier paper [2], where it is shown that in the case where $f(z)$ is a *polynomial* and $k = 1$ the 1-shifted sequence $\{z^n \cdot f(z)\}_{n=0}^\infty$ is a basic sequence in $W(D)$ if and only if the zeros of $f(z)$ on the circle $|z| = 1$ are all simple. As the nature of this result suggests, the treatment of the general case where $f(z)$ is arbitrary, even when $k = 1$, is not immediate. Using the observations about k -shifted sequences made in §2, together with the techniques of [2], one can give a characterization of the basicity of k -shifted sequences $\{z^{kn} \cdot f(z)\}_{n=0}^\infty$ when $f(z)$ is a polynomial. This and other related results concerning shift sequences will be given in a forthcoming paper.

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