GEOMETRIC CONDITIONS FOR INTERPOLATION

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ABSTRACT. Let $\{z_n\}$ be a sequence lying in either the upper half-plane or the unit disc in the complex plane. If $\{z_n\}$ is a separated sequence we give a simple geometric condition that implies the sequence is an interpolating sequence for the algebra of bounded holomorphic functions. This result contains most of the known results of this type.

0. Introduction. Let Δ denote the unit disc $|z| < 1$ in the complex plane and $\partial\Delta$ its boundary. A sequence $\{z_n\}$ in Δ is an *interpolating sequence* for the algebra $H^\infty(\Delta)$ of bounded holomorphic functions in Δ if, for each bounded sequence of complex numbers $\{w_n\}$, there exists a function f in $H^\infty(\Delta)$ such that $f(z_n) = w_n$ for all n . For $\{z_n\}$ to be interpolating it is necessary and sufficient that

$$(0.1) \quad \inf_n \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \chi(z_n, z_k) > 0,$$

where

$$\chi(z, w) \equiv |(z - w)/(1 - \bar{w}z)|$$

is the pseudohyperbolic distance on Δ . The necessity was established independently by L. Carleson [1], W. K. Hayman [5], and D. J. Newman [7]; the sufficiency was proved by Carleson [1]. A sequence $\{z_n\}$ that satisfies (0.1) is said to be *uniformly separated*.

In practice, condition (0.1) is difficult to work with. Thus, several authors have given more accessible conditions for interpolation in the particular case that $\{z_n\}$ converges to a boundary point. See Garnett [3, pp. 284–301, 438–439] for some of these results and a general treatment of interpolating sequences. Some involve a separation condition that is weaker than (0.1). The sequence $\{z_n\}$ is called *pairwise separated* or just *separated* if it satisfies

$$(0.2) \quad \inf_{n \neq m} \chi(z_n, z_m) > 0.$$

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D. H. Wortman [9] proved the following equivalence concerning sequences $\{z_n\}$ that lie on a convex boundary arc and satisfy

$$(0.3) \quad \inf_n \chi(z_n, z_{n+1}) > 0.$$

A sequence $\{z_n\}$ that satisfies (0.3) is said to be *consecutively separated*.

THEOREM A. *Let Γ be a convex curve in $\Delta \cup \partial\Delta$ with $\Gamma \cap \partial\Delta = \{1\}$ and Γ tangent to $\partial\Delta$ at 1. If $\{z_n\}$ is a sequence of points on $\Gamma \setminus \{1\}$ with $|z_n| \uparrow 1$, then (0.1) is satisfied if and only if $\{z_n\}$ is consecutively separated.*

Another characterization is due to M. L. Weiss [8]; this result contains the previous result of Wortman. (In his paper [8], Weiss was mainly concerned with sequences that lie on special curves in Δ that are tangent to $\partial\Delta$ at 1. In his statement of the following theorem, he included the hypothesis that $\{z_n\} \rightarrow 1$ tangentially; however, this condition is readily seen to be extraneous.)

THEOREM B. *If the sequence $\{z_n = r_n e^{i\theta_n}\}$ in Δ satisfies $r_n \uparrow 1$ and $\theta_n \downarrow 0$, then $\{z_n\}$ satisfies (0.1) if and only if $\{z_n\}$ is consecutively separated.*

Recently, E. A. Gerber and M. L. Weiss [4, Theorem 4.6] established the following supplement to Theorem B.

THEOREM C. *If the sequence $\{z_n = r_n e^{i\theta_n} = x_n + iy_n\}$ in Δ is consecutively separated with $\theta_n \downarrow 0$ and $x_n \uparrow 1$, then it is interpolating.*

Our main result, Theorem 1, contains Theorems A, B and C. To begin, we define a cone $\mathcal{C}$ with vertex angle α to be a closed region of the form $\{re^{i\theta} : \theta_0 \leq \theta \leq \theta_0 + \alpha, r \geq 0\}$ ($\pi/2 < \alpha < \pi$). The rays $\{\theta = \theta_0\}$ and $\{\theta = \theta_0 + \alpha\}$ are referred to, respectively, as the *right* and *left boundary rays* of $\mathcal{C}$. The cone that is the image of $\mathcal{C}$ under $T(z) = -z$ is denoted by $-\mathcal{C}$, and the cone that is the image of $\mathcal{C}$ under $T(z) = z + \zeta$ is denoted by $\mathcal{C}(\zeta)$. Also, the right and left boundary rays of $\mathcal{C}(\zeta)$ are the respective translated images of the right and left boundary rays of $\mathcal{C}$.

We now state our main results. In §1 we shall prove Theorem 1, and in §2 we show that Corollary 1 follows readily from it. In the statements of our results, we shall allow $\{z_n\}$ to lie either in Δ or in the open upper half-plane $\mathcal{H}$. In $\mathcal{H}$ the pseudohyperbolic metric takes the form $\chi(z, w) \equiv |(z - w)/(z - \bar{w})|$, and the definition for the sequence to be interpolating is the same as that in Δ with $H^\infty(\Delta)$ replaced with $H^\infty(\mathcal{H})$.

THEOREM 1. *Let $\{z_n\}$ be a separated sequence that lies in either Δ or $\mathcal{H}$, α a number strictly between $\pi/2$ and π , and N a nonnegative integer. If to each n there corresponds a cone $\mathcal{C}_n$ with vertex angle α such that*

$$(0.4) \quad z_m \in \mathcal{C}_n(z_n) \quad \text{for all but at most } N \text{ indices } m > n,$$

then $\{z_n\}$ is an interpolating sequence.

COROLLARY 1. Let $\{z_n\}$ be a consecutively separated sequence lying in either Δ or $\mathcal{H}$. If there exists a cone $\mathcal{C}$ such that

$$(0.5) \quad z_{n+1} \in \mathcal{C}(z_n) \quad \text{for all } n,$$

then $\{z_n\}$ is an interpolating sequence.

It is clear that Theorems A, B and C are immediate consequences of Corollary 1.

1. Proof of Theorem 1. Choose an even integer J such that $2\pi/J < (\pi - \alpha)/2$, and let $\mathcal{C}^j$ ($j = 1, 2, \dots, J$) be all the cones with vertex angle $\pi - 2\pi/J$ formed from the rays $\{\theta = 2\pi j/J\}$ ($j = 1, 2, \dots, J$). Then clearly each cone $\mathcal{C}_n$ is contained in one of the cones $\mathcal{C}^j$. Therefore, since the finite union of interpolating sequences is also interpolating, provided the union is a separated sequence, it suffices to prove Theorem 1 with hypothesis (0.4) replaced with the hypothesis that there exists a fixed cone $\mathcal{C}$ with vertex angle α such that

$$(1.1) \quad z_m \in \mathcal{C}(z_n) \quad \text{for all but at most } N \text{ indices } m > n.$$

We prove Theorem 1 first for $\{z_n\}$ in $\mathcal{H}$. For $z, w \in \mathcal{H}$,

$$(1.2) \quad \frac{\chi(z, w)}{1 + \chi(z, w)} \leq \frac{|z - w|}{2 \operatorname{Im}(z)} \leq \frac{\chi(z, w)}{1 - \chi(z, w)}.$$

To see this, note that

$$\frac{|z - w|}{2 \operatorname{Im}(z)} = \frac{|z - w|}{|(z - \bar{w} + \bar{w} - \bar{z})|} = \frac{|(z - w)/(z - \bar{w})|}{|1 + (\bar{w} - \bar{z})/(z - \bar{w})|},$$

and then apply the triangle inequalities to the denominator of the last term. By the left inequality in (1.2) and the fact that $\{z_n\}$ is separated, there exists a constant A such that

$$(1.3) \quad \operatorname{Im}(z_n) \leq A |z_n - z_m| \quad \text{for all } n \text{ and } m.$$

We shall make use of the inequality later in the proof.

According to J. Garnett [2] (see also [6, pp. 280–282]), to prove that the separated sequence $\{z_n\}$ is interpolating it is sufficient to show that it generates a Carleson measure, that is, to show that there exists a constant K such that $\sum_{z_n \in \mathcal{S}} \operatorname{Im}(z_n) < Kd$ for each square $\mathcal{S}$ of the form $\{x + iy: x_0 \leq x \leq x_0 + d, 0 \leq y \leq d\}$.

Consider an arbitrary square $\mathcal{S} = \{x + iy: x_0 \leq x \leq x_0 + d, 0 \leq y \leq d\}$. We may assume that the vertex angle α of $\mathcal{C}$ is close enough to π so that $\mathcal{S} \subset \mathcal{C}(z)$ for one of the corners z of $\mathcal{S}$. We give the proofs for $z = x_0 + id$; the other three proofs are similar. See Figure 1.

Let $\mathcal{Q}$ denote the parallelogram obtained by intersecting $\mathcal{C}(x_0 + id)$ and $-\mathcal{C}(x_0 + d)$. Clearly, $\mathcal{S} \subset \mathcal{Q}$, and to complete the proof we need only show that there exists a constant K , independent of x_0 and d , such that

$$(1.4) \quad \sum_{z_n \in \mathcal{Q}} \operatorname{Im}(z_n) \leq Kd.$$

For $z_n \in P$, let $m(n)$ be the smallest $m > n$ such that $z_m \in \mathcal{C}(z_n) \cap \mathcal{Q}$. Then let R_n and L_n denote the intersection of $\mathcal{C}(z_n) \setminus \mathcal{C}(z_{m(n)})$ with R and L , respectively. (See Figure 1.) Note that if $z_{m(n)} \in \partial \mathcal{C}(z_n)$, then one of R_n or L_n equals $\emptyset$. In any case it is easy to see that $|z_n - z_{m(n)}| \leq |R_n| + |L_n|$, and by (1.3) we have

$$(1.8) \quad A^{-1} \sum_{z_n \in P} \text{Im}(z_n) \leq \sum_{z_n \in P} (|R_n| + |L_n|).$$

We claim that no $z \in L$ is contained in more than $2N$ of the segments L_n . Fix $z \in L$ and let n^* be the smallest n such that $z \in L_n$. If $m \geq m(n^*)$ and $z_m \in \mathcal{C}(z_{m(n^*)})$, then $z \notin L_m$. Consequently, (1.1) implies there are at most N indices $m \geq m(n^*)$ for which $z \in L_m$. By the same hypothesis $m(n^*) - n^* \leq N + 1$. This establishes the claim and, hence, it follows that

$$(1.9) \quad \sum_{z_n \in P} |L_n| \leq (2N) |L|.$$

A similar argument shows that

$$(1.10) \quad \sum_{z_n \in P} |R_n| \leq (2N) |R|.$$

Using inequalities (1.8)–(1.10) together with (1.5) we obtain

$$(1.11) \quad \sum_{z_n \in P} \text{Im}(z_n) \leq 2NABd.$$

In view of (1.7) and (1.11) we see that (1.4) holds for K equal to the sum of $2NAB$ and $(N + 1)B$. This completes the proof of Theorem 1 for $\{z_n\}$ in $\mathcal{K}$.

We now prove Theorem 1 for $\{z_n\}$ in Δ . Because $z_m \in \mathcal{C}(z_n)$ implies $\mathcal{C}(z_m) \subset \mathcal{C}(z_n)$, a simple geometric argument can be used to deduce that $\{z_n\}$ converges to a point $\zeta \in \partial \Delta$. Using a rotation if necessary, we may assume that $\zeta = 1$. Then, since $1 \in \mathcal{C}(z_n)$ for each n and since $\alpha < \pi$, there is a $\tau \in \partial \Delta$ with $\tau \notin \mathcal{C}(z_n)$ for n large; hence we may assume that τ belongs to none of the cones $\mathcal{C}(z_n)$.

Consider the conformal mapping

$$T(z) = e^{i\theta}(z - 1)/(z - \tau)$$

of Δ onto $\mathcal{K}$. Then $\{T(z_n)\}$ is a separated sequence in $\mathcal{K}$ that converges to 0. Since $T(\tau) = \infty$, the image of $\mathcal{C}(z_n) \cap \Delta$ is the intersection with $\mathcal{K}$ of the interiors of two bounded circles that intersect at $T(z_n)$ and $T(\infty)$. Thus if $\mathcal{C}_n(T(z_n))$ is the cone that contains 0 and is formed by the tangents to the intersecting circles at $T(z_n)$, then this cone has a vertex angle α and $T(z_m) \in \mathcal{C}_n(T(z_n))$ for all but at most N indices $m > n$. Therefore, by Theorem 1 for sequences in $\mathcal{K}$, the sequence $\{T(z_n)\}$ is an interpolating sequence in $\mathcal{K}$. This clearly implies that $\{z_n\}$ is an interpolating sequence in Δ , and the proof of Theorem 1 is complete.

2. Proof of Corollary 1. Suppose $\{z_n\}$ is a consecutively separated sequence in $\mathcal{K}$ and suppose condition (0.5) is satisfied for some cone $\mathcal{C}$. Then a basic geometric argument will show that the distance between z_n and any point in $\mathcal{C}(z_{n+1})$ is at least $|z_n - z_{n+1}| \sin \alpha$, where α is the vertex angle of $\mathcal{C}$. Thus, for any $m > n$ we have

$$(2.1) \quad |z_n - z_m| \geq |z_n - z_{n+1}| \sin \alpha.$$

The left inequality in (1.2) for $m = n + 1$ gives $|z_n - z_{n+1}|/y_n \geq \chi(z_n, z_{n+1})$, where $y_n = \text{Im}(z_n)$. Combining this inequality with (2.1) and using the hypothesis that $\{z_n\}$ is consecutively separated, we obtain a constant Q such that

$$|z_n - z_m|/y_n > Q \quad \text{for } m \neq n.$$

Now we deduce from the right inequality in (1.2) that $\chi(z_n, z_m) > Q/(Q + 2)$; that is, $\{z_n\}$ is separated, and it follows from Theorem 1 that $\{z_n\}$ is interpolating.

The proof for $\{z_n\}$ in Δ is identical once we obtain the disc analogue of (1.2): for $z, w \in \Delta$,

$$(2.2) \quad \frac{\chi(z, w)}{1 + |z|\chi(z, w)} \leq \frac{|z - w|}{1 - |z|^2} \leq \frac{\chi(z, w)}{1 - |z|\chi(z, w)}.$$

The proof is entirely similar to that of (1.2) in that

$$\frac{|z - w|}{1 - |z|^2} = \frac{|z - w|}{|1 - z\bar{w} + z\bar{w} - z\bar{z}|} = \frac{|(z - w)/(1 - z\bar{w})|}{|1 + z((\bar{w} - \bar{z})/(1 - z\bar{w}))|}.$$

The triangle inequalities applied to the denominator of the last term give (2.2).

3. Sharpness of Theorems 1 and C. Let p be a positive integer and consider the square

$$\mathfrak{S}_p \equiv \{x + iy: 0 \leq x \leq 2^{-p^2}, 0 \leq y \leq 2^{-p^2}\}.$$

Within $\mathfrak{S}_p$ consider the set of points

$$\mathfrak{s}_p \equiv \{(\sigma + i)2^{-p^2-\lambda}: \lambda = 0, 1, \dots, p-1; \sigma = 1, 2, \dots, 2^\lambda\},$$

and set $\mathfrak{s} \equiv \bigcup \mathfrak{s}_p$. Then the following facts are easily proven:

- (I) $\sum_{z \in \mathfrak{s}_p} \text{Im}(z) = p2^{-p^2}$;
- (II) $\chi(z, w) \geq \frac{1}{3}$ for any distinct points $z, w \in \mathfrak{s}$; and
- (III) $\mathfrak{s}_p \subset \{|z - ia_p| \leq a_p\}$ for $a_p = 2^{-p^2+p-2} + 2^{-p^2-p}$.

Thus, by ordering $\mathfrak{s}$ from "top down and left to right", we obtain the following example.

EXAMPLE. *There exists a noninterpolating, separated Blaschke sequence $\{z_n\} \subset \mathcal{H}$ such that $\{z_n\}$ is eventually in every disc in $\mathcal{H}$ that is tangent to $\partial\mathcal{H}$ at 0 and to each z_n there corresponds a cone $\mathcal{C}_n$ in the complement of $\mathcal{H}$ with the property that $z_m \in \mathcal{C}_n(z_n)$ for each $m > n$. (Note that each $\mathcal{C}_n$ has vertex angle $\alpha_n < \pi$, but $\sup_n \alpha_n = \pi$.)*

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