NONUNIQUENESS OF TEICHMÜLLER EXTREMAL MAPPINGS

EDGAR REICH¹

ABSTRACT. A boundary correspondence for the unit disk is found with the following property. Among the extremal quasiconformal mappings of the disk with the specified boundary correspondence there are at least two Teichmüller mappings.

1. Introduction. Our intention will be to show that the heretofore open question in the theory of plane qc (quasiconformal) mappings, as to whether more than one Teichmüller extremal mapping with a specified boundary correspondence can exist, has an affirmative answer. Precisely what we mean by a Teichmüller extremal mapping is explained below.

Let Ω and $\tilde{\Omega}$ be simply connected domains of hyperbolic type in the complex z and w planes, respectively. For the purpose of this discussion we shall say that a qc mapping $w = f(z)$ of Ω onto $\tilde{\Omega}$ is a *Teichmüller mapping*, or $f \in T_k(\Omega)$, for short, if the complex dilatation $\kappa(z) = f_{\bar{z}}/f_z$ has the form

$$\kappa(z) = k \overline{\varphi(z)} / |\varphi(z)|, \quad z \in \Omega,$$

where k is a constant, $0 < k < 1$, and $\varphi(z)$ is meromorphic² in Ω . This definition is invariant in the following sense. If $F: \Omega_0 \rightarrow \Omega$ and $G: \tilde{\Omega}_0 \rightarrow \tilde{\Omega}$ are conformal, then $f_0 = G^{-1} \circ f \circ F \in T_k(\Omega_0)$, with corresponding meromorphic function $\varphi_0(z) = F'(z)^2 \varphi(F(z))$.

Let h be a given homeomorphism of $\partial\Omega$ onto $\partial\tilde{\Omega}$ in terms of prime ends. We postulate that the class Q_h of qc mappings g of Ω onto $\tilde{\Omega}$ with boundary values h , $h = g|_{\partial\Omega}$, is nonempty. Let $K[g]$, $g \in Q_h$, denote the maximal dilatation of g , and let $K_h^* = \inf\{K[g], g \in Q_h\}$ denote the extremal dilatation corresponding to h . If $f \in Q_h$ and $K[f] = K_h^*$, then f is an *extremal mapping* for h . A normality argument guarantees that Q_h contains at least one extremal mapping.

If f is an extremal mapping for h and $f \in Q_h \cap T_k(\Omega)$, we say that f is a *Teichmüller extremal* for h . It is, of course, then necessarily the case that $k = (K_h^* - 1)/(K_h^* + 1)$. Since f is a Teichmüller extremal if and only if f_0 is a Teichmüller extremal, there is, in principle, no loss of generality if all qc mappings were normalized to, say, self-mappings of the unit disk; however, for the construction of examples, such a normalization is cumbersome, as the auxiliary conformal

Received by the editors September 3, 1982.

1980 *Mathematics Subject Classification*. Primary 30C75.

¹Work done with support by National Science Foundation grant MCS 8100760.

²Note that we do not, as is often the case, restrict φ to be holomorphic, but we allow it to have poles in Ω .

mappings only tend to obscure what is going on. The special interest in Teichmüller extremals is due, firstly, to the still open question whether Q_h necessarily contains at least one Teichmüller extremal and, secondly, to the classical fact that if a Teichmüller extremal f does exist in Q_h , and if $\varphi(z)$ is holomorphic in Ω and $\varphi \in \mathcal{L}^1(\Omega)$, then Q_h contains no other extremal mapping. In a recent paper [2] Strebel has succeeded in constructing a boundary correspondence h_0 of the unit circle onto itself, for which he proves that one of the following two alternatives holds: *Either* (i) Q_{h_0} contains no Teichmüller extremals at all, *or* (ii) Q_{h_0} contains infinitely many Teichmüller extremals. In view of the prevalent conjecture that alternative (i) is impossible, Strebel's result suggests the possibility that several Teichmüller extremals could exist for some boundary correspondences h . This is what we shall in fact prove. Specifically we show the following

THEOREM. *For a parabolic region Ω there exists, in addition to the horizontal stretch f_1 parallel to the axis of the parabola, another Teichmüller extremal f_2 with the same boundary values $h = f_1|_{\partial\Omega} = f_2|_{\partial\Omega}$.*

2. Proof of the Theorem. We choose for Ω the parabolic region

$$(2.1) \quad \Omega = \Omega_1 = \{z = x + iy: 4(x+1) \geq y^2, x > -1\};$$

that is, the vertex is at $z = -1$ and the focus at $z = 0$. Given $K > 1$, we define $f_1(z)$, $z \in \Omega_1$, by

$$(2.2) \quad w = u + iv = Kx + K - K^{-1} + iy.$$

Of course, h is then given by (2.2), except that z is restricted to lie on $\partial\Omega_1$. It is clear that $f_1 \in T_k(\Omega_1)$; in fact the complex dilatation of f_1 is

$$\kappa_1(z) = k \overline{\varphi_1(z)} / |\varphi_1(z)|, \quad \text{with } k = (K-1)/(K+1), \varphi_1(z) \equiv 1.$$

According to a result first proved by E. Blum [1] for affine stretches of the parabola, $K^*_h = K$; that is, f_1 is a Teichmüller extremal for h .

To prepare for the construction of f_2 we need to refer to some elementary analytical facts about parabolas.

For $a > 0$, let

$$\Omega_a = \{z = x + iy: 4a(x+a) \geq y^2, x > -a\}.$$

$\partial\Omega_a$ is a parabola with vertex at $z = -a$ and focus at $z = 0$. Since $z = x + iy = (y^2 - 4a^2)/4a + iy$ whenever $z \in \partial\Omega_a$, the curve $\partial\Omega_a$ satisfies the differential equation

$$(2.3) \quad (dz/dy)^2 = z/a.$$

Conversely, the trajectories for which $dz^2/z \geq 0$ ($z \neq 0$) consist of the family of parabolas $\{\partial\Omega_a\}$, $a > 0$, together with the positive real axis.

The additive constant $K - 1/K$ in the definition (2.2) of h has been chosen in such a manner that the parabolic region $\tilde{\Omega}_1 = f_1(\Omega)$ has its focus at $w = 0$; as is readily verified,

$$(2.4) \quad \tilde{\Omega}_1 = \Omega_p, \quad p = 1/K.$$

The following very simple, but qm (quasimiraculous), fact is at the heart of the construction of f_2 .

LEMMA. $w = h(z)$ be the mapping of $\partial\Omega_1$ onto $\partial\tilde{\Omega}_1$ defined by (2.2). Then

$$(2.5) \quad dw^2/w = K(dz^2/z), \quad z \in \partial\Omega_1.$$

PROOF. By (2.3) the curve $\partial\Omega_1$ satisfies

$$(2.6) \quad (dz/dv)^2 = z.$$

Similarly, by (2.3), the curve $\partial\tilde{\Omega}_1$ satisfies

$$(2.7) \quad (dw/dv)^2 = Kw.$$

But, for the boundary correspondence h , $v = y$. Dividing (2.7) by (2.6) we therefore obtain

$$(dw/dz)^2 = Kw/z.$$

In formal quadratic differential notation this becomes (2.5). $\square$

Let

$$\Omega^+ = \{z \in \Omega_1 : \operatorname{Im} z > 0\}, \quad \tilde{\Omega}^+ = \{w \in \tilde{\Omega}_1 : \operatorname{Im} w > 0\}.$$

In Ω^+ , let $\zeta = \Phi(z)$ be the branch of $\sqrt{z}$ with $\operatorname{Re} \Phi(z) > 0$, and in $\tilde{\Omega}^+$, let $\lambda = \Psi(w)$ be the branch of $\sqrt{w}$ with $\operatorname{Re} \Psi(w) > 0$. Since, by (2.3),

$$(2.8) \quad d\zeta^2 = dz^2/4z \geq 0, \quad z \in \partial\Omega_1,$$

and

$$(2.9) \quad d\lambda^2 = dw^2/4w \geq 0, \quad w \in \partial\tilde{\Omega}_1,$$

we see that $\Phi(\Omega^+)$ and $\Psi(\tilde{\Omega}^+)$ are half-strips:

$$\Phi(\Omega^+) = \{\zeta = \xi + i\eta : \xi > 0, 0 < \eta < 1\},$$

$$\Psi(\tilde{\Omega}^+) = \{\lambda = \sigma + it : \sigma > 0, 0 < t < 1/\sqrt{K}\}.$$

Moreover, in view of (2.5), (2.8) and (2.9), we conclude that the homeomorphism h between $\partial\Omega_1$ and $\partial\tilde{\Omega}_1$ induces the mapping

$$(2.10) \quad d\sigma^2 = K d\xi^2$$

between the upper boundary $\{\xi + i\eta : \xi \geq 0, \eta = 1\}$ of $\Phi(\Omega^+)$ and the upper boundary $\{\sigma + it : \sigma > 0, t = 1/\sqrt{K}\}$ of $\Psi(\tilde{\Omega}^+)$. Since $\lambda = i/\sqrt{K}$ corresponds to $\zeta = i$ under h , (2.10) implies that h induces the boundary correspondence $\sigma = \sqrt{K}\xi$, for $\xi \geq 0, \eta = 1$. Let $\lambda = A_K(\zeta)$ be the horizontal stretch by the factor K , uniformly rescaled by the factor $\sqrt{K}$:

$$\lambda = \sigma + it = A_K(\zeta) = \sqrt{K}\xi + (i/\sqrt{K})\eta, \quad \zeta = \xi + i\eta \in \Phi(\Omega^+),$$

and let us define

$$(2.11) \quad f_2(z) = \Psi^{-1} \circ A_K \circ \Phi(z), \quad z \in \Omega, \operatorname{Im} z \geq 0,$$

$$(2.12) \quad f_2(z) = \overline{f_2(\bar{z})}, \quad z \in \Omega, \operatorname{Im} z < 0.$$

Our construction automatically guarantees that f_2 is a K -qc mapping of Ω_1 onto $\tilde{\Omega}_1$, with $f_2|_{\partial\Omega_1} = h = f_1|_{\partial\Omega_1}$, as required. From (2.11) we find that the complex dilatation $\kappa_2(z)$ of $f_2(z)$ is

$$(2.13) \quad \kappa_2(z) = k \frac{\overline{\varphi_2(z)}}{|\varphi_2(z)|}, \quad \varphi_2(z) = \Phi'(z)^2 = \frac{1}{4z}, \quad k = \frac{K-1}{K+1},$$

when $z \in \Omega^+$. On the other hand, (2.12) implies that $\kappa_2(z) = \overline{\kappa_2(\bar{z})}$. Hence (2.13) holds in all of Ω_1 . Thus, $f_2 \in T_k(\Omega_1)$. Since $K[f_2] = K = K_h^*$, it is therefore clear that f_2 is a Teichmüller extremal mapping for h . The fact that f_1 and f_2 are different is, of course, trivial, e.g., because κ_1 and κ_2 are different.

REFERENCES

1. Eugen Blum, *Die Extremalität gewisser Teichmüllerscher Abbildungen des Einheitskreises*, Comment. Math. Helv. **44** (1969), 319–340.
2. Kurt Strebel, *Is there always a unique extremal Teichmüller mapping?* (preprint).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455