EMBEDDING $l_p^{n^\alpha}$ IN $l_{p,q}^n$

N. L. CAROTHERS AND P. H. FLINN

ABSTRACT. It is shown that for $1 < p < \infty$, $1 \leq q \leq \infty$ and $0 < \alpha < 1$, there is a constant $C = C(p, q, \alpha) < \infty$ such that l_p^k is C -isomorphic to a subspace of $l_{p,q}^n$ where $k = O(n^\alpha)$.

L. Tzafriri first proved (unpublished) that if X is a Banach space with 1-symmetric basis, then for some $1 \leq p \leq \infty$, X contains uniformly isomorphic copies of l_p^k , for all $k \geq 1$, spanned by disjointly supported vectors. (J. L. Krivine has since provided a stronger result [8].) From Tzafriri's techniques it follows that the Lorentz sequence space $l_{p,q}$ contains a copy of l_p^k on disjoint vectors for all k . The proof, however, is not quantitative and gives only limited information concerning the finite-dimensional question: how large may k be taken so that $l_{p,q}^n$ contains l_p^k on disjoint vectors? In this note, using a modification of Tzafriri's ideas, we construct a copy of l_p^k on disjoint vectors in $l_{p,q}^n$ and show that k may be taken to be as large as n^α for any fixed $0 < \alpha < 1$ and all n . This result also appears, in somewhat different forms, in [3, and 4].

For $1 \leq q \leq p < \infty$ and $n \geq 1$ we take $l_{p,q}^n$ to be $\mathbf{R}^n$ with its usual basis $(e_i)_{i=1}^n$ and supplied with the norm

$$(1) \quad \left\| \sum_{i=1}^n x_i e_i \right\|_{p,q} = \left\{ \sum_{i=1}^n x_i^{*q} (i^{q/p} - (i-1)^{q/p}) \right\}^{1/q},$$

where $(x_i^*)_{i=1}^n$ is the decreasing rearrangement of $(|x_i|)_{i=1}^n$. For $1 < p < q \leq \infty$ we take $l_{p,q}^n$ to be the dual of $l_{p',q'}^n$, where $1/p + 1/p' = 1 = 1/q + 1/q'$, and give $l_{p,q}^n$ the usual dual norm, which we denote by $\| \cdot \|_{p,q}$. If $1 < p < q \leq \infty$, (1) defines a quasinorm on $l_{p,q}^n$ which is equivalent to $\| \cdot \|_{p,q}$; in particular, there is a constant $A = A(p, q) < \infty$ such that

$$(2) \quad A^{-1} \|x\|_{p,q} \leq \|x\|_{p,q} \leq A \|x\|_{p,q}, \quad x \in \mathbf{R}^n$$

(cf. [1 or 6]). Of course, $l_{p,p}^n \equiv l_p^n$ for any p . If $1 \leq q \leq p < \infty$, and $(u_j)_{j=1}^k$ are disjointly supported vectors in $l_{p,q}^n$, it is known that

$$(3) \quad \left\| \sum_{j=1}^k u_j \right\|_{p,q} \geq \left(\sum_{j=1}^k \|u_j\|_{p,q}^p \right)^{1/p}$$

(cf., e.g., [2 or 9, Example 1.f.19]).

Received by the editors August 30, 1982. Presented to the Society at the Norman, Oklahoma, Meeting, March 19, 1983.

1980 *Mathematics Subject Classification*. Primary 46B99.

Key words and phrases. Lorentz sequence space, basic sequence.

©1983 American Mathematical Society
0002-9939/82/0000-2026/\$02.00

Fix $n \geq 1$, $1 \leq q < p < \infty$ and $0 < \alpha < 1$. We set $m = [n^{1-\alpha}]$, $k = [n^\alpha]$ and define vectors $(v_j)_{j=1}^k$ in $l_{p,q}^n$ by

$$v_j = \sum_{i=1}^m i^{-1/p} e_{i+(j-1)m}, \quad j = 1, \dots, k.$$

Using (1) and the Mean-Value Theorem it is easy to see that

$$(4) \quad (q/p) \log(m+1) \leq \|v_1\|_{p,q}^q \leq 1 + (q/p)(1 + \log m).$$

In order to prove the result stated in the abstract we will show that $(v_j)_{j=1}^k$ span a well-complemented copy of l_p^k in $l_{p,q}^n$, $1 \leq q < p < \infty$ (the case $p < q$ will then follow by duality). Using this notation, we first prove

PROPOSITION 1. *Given $1 \leq q < p < \infty$ and $0 < \alpha < 1$, there is a constant $C = C(p, q, \alpha) < \infty$ such that*

$$(5) \quad \left\| \sum_{j=1}^k a_j v_j \right\|_{p,q} \leq C \|v_1\|_{p,q} \left(\sum_{j=1}^k |a_j|^p \right)^{1/p}$$

for all scalars $(a_j)_{j=1}^k$.

PROOF. If we set $v = \sum_{i=1}^n i^{-1/p} e_i$, it follows from (4) that there is a constant $C = C(p, q, \alpha) < \infty$ such that $\|v\|_{p,q} \leq C \|v_1\|_{p,q}$ for all n sufficiently large. Consequently, it suffices to show that if $a_1 \geq a_2 \geq \dots \geq a_k \geq 0$ with $\sum_{j=1}^k a_j^p = 1$, then $\|\sum_{j=1}^k a_j v_j\|_{p,q} \leq \|v\|_{p,q}$. Indeed, fix $1 \leq i \leq n$ and for each $1 \leq j \leq k$ let m_j be the largest integer for which $1 \leq m_j \leq m$ and $a_j m_j^{-1/p} \geq i^{-1/p}$. Then $a_j^p i \geq m_j$, and hence $i \geq \sum_{j=1}^k m_j$. That is, the i th coordinate of the decreasing rearrangement of $\sum_{j=1}^k a_j v_j$ does not exceed the i th coordinate of v , completing the proof. $\square$

PROPOSITION 2. *Given $1 \leq q < p < \infty$ and $0 < \alpha < 1$, there is a constant $D = D(p, q, \alpha) < \infty$ such that the span of $(v_j)_{j=1}^k$ is the range of a projection Q on $l_{p,q}^n$ with $\|Q\| \leq D$.*

PROOF. Let $1/p + 1/p' = 1 = 1/q + q/q'$ and define vectors $(v'_j)_{j=1}^k$ in $l_{p',q'}^n \equiv (l_{p,q}^n)^*$ by

$$v'_j = \sum_{i=1}^m i^{-1/p'} e_{i+(j-1)m}, \quad j = 1, \dots, k.$$

It follows from (1), (2) and (4) that there is a constant $B = B(p, q) < \infty$ such that $\|v_1\|_{p,q} \|v'_1\|_{p',q'} \leq B v'_1(v_1)$, for all n sufficiently large. Now define Q on $l_{p,q}^n$ by

$$Qx = (v'_1(v_1))^{-1} \sum_{j=1}^k v'_j(x) v_j, \quad x \in l_{p,q}^n.$$

Obviously, Q is a projection onto the span of $(v_j)_{j=1}^k$. And if $C = C(p, q, \alpha) < \infty$ is the constant from Proposition 1, then using (5) and (3) we have

$$\begin{aligned} \|Qx\|_{p,q} &\leq C \|v_1\|_{p,q} (v'_1(v_1))^{-1} \left(\sum_{j=1}^k |v'_j(x)|^p \right)^{1/p} \\ &\leq BC \left(\sum_{j=1}^k \left\| \sum_{i=(j-1)m+1}^{jm} x_i e_i \right\|_{p,q}^p \right)^{1/p} \\ &\leq BC \|x\|_{p,q}. \quad \square \end{aligned}$$

We conclude by giving an upper estimate on the dimension of complemented l_p -subspaces of $l_{p,q}^n$. More precisely, we have

PROPOSITION 3. *Let $A: l_p^k \rightarrow l_{p,q}^n$ and $B: l_{p,q}^n \rightarrow l_p^k$ be operators for which BA is the identity on l_p^k . Then there is a constant $C = C(p, q) < \infty$ such that $k \leq C \|A\| \|B\| n / (\log n)^{1/p-1/q}$.*

PROOF. First recall that for a finite-dimensional space E the L_p -factorization constant $\gamma_p(E)$ is defined by $\gamma_p(E) = \inf \|S\| \|T\|$, where the infimum is taken over all $S: L_p[0, 1] \rightarrow E$ and $T: E \rightarrow L_p[0, 1]$ for which ST is the identity on E .

Given A and B as in the statement of the proposition, a result due to Gordon and Lewis (cf., e.g., [5]) shows that $\gamma_p(l_{p,q}^n) \leq nk^{-1} \|A\| \|B\|$; thus it only remains to compute a lower bound for $\gamma_p(l_{p,q}^n)$. By duality, it suffices to consider the case $p \geq 2$; we prove only the case $q \leq p$, the proof in the other case being identical. Now it follows from the proof of Theorem 1.d.7 in [9] that there is a constant D , depending only on p , such that

$$(6) \quad \left(\sum_{i=1}^m \|x_i\|_{p,q}^p \right)^{1/p} \leq D \cdot \gamma_p(l_{p,q}^n) \cdot \left\| \left(\sum_{i=1}^m |x_i|^p \right)^{1/p} \right\|_{p,q},$$

for all $x_1, \dots, x_m$ in $l_{p,q}^n$. An appropriate choice of the x_i 's (suggested by Example 1.f.19 in [9]) provides the necessary lower bound for $\gamma_p(l_{p,q}^n)$. For $1 \leq i \leq n$ let τ_i be the cyclic permutation of $\{1, 2, \dots, n\}$ defined by $\tau_i(j) = (i+j) \bmod n$, and define x_i in $l_{p,q}^n$ by

$$x_i = \sum_{j=1}^n (\tau_i(j))^{-1/p} e_j.$$

Then,

$$\left(\sum_{i=1}^n |x_i|^p \right)^{1/p} = \sum_{j=1}^n \left(\sum_{i=1}^n \tau_i(j)^{-1} \right)^{1/p} e_j = \left(\sum_{i=1}^n i^{-1} \right)^{1/p} \sum_{j=1}^n e_j,$$

and so it follows from (4) and (6) that there is a constant $C = C(p, q)$ such that

$$(7) \quad n^{1/p} (\log n)^{1/q} \leq C \cdot \gamma_p(l_{p,q}^n) \cdot n^{1/p} (\log n)^{1/p},$$

which finally implies that

$$k \leq (\gamma_p(l_{p,q}^n))^{-1} \cdot n \cdot \|A\| \|B\| \leq C \cdot n \cdot \|A\| \|B\| (\log n)^{1/p-1/q}. \quad \square$$

REMARK. The formal identity (and Hölder's inequality) easily gives $\gamma_p(l_{p,q}^n) \leq C(p, q)(\log n)^{|1/p-1/q|}$, and so from (7) $\gamma_p(l_{p,q}^n)$ is actually equivalent to $(\log n)^{|1/p-1/q|}$ (up to constants depending only on p and q). In fact, this same observation will also show that the Banach-Mazur distance $d(l_{p,q}^n, l_{p,q}^n)$ and the p -concavity constant $M_p(l_{p,q}^n)$ are likewise equivalent to $(\log n)^{|1/p-1/q|}$.

REFERENCES

1. G. D. Allen, *Duals of Lorentz spaces*, Pacific J. Math. **77** (1978), 287–291.
2. Z. Altshuler, *The modulus of convexity of Lorentz and Orlicz sequence spaces*, Notes in Banach Spaces (H. E. Lacey, ed.), Univ. of Texas Press, 1980.
3. N. L. Carothers, Dissertation, Ohio State Univ., 1982.
4. P. H. Flinn, Dissertation, Ohio State Univ., 1981.
5. Y. Gordon and S. Reisner, *Some geometric properties of Banach spaces of polynomials*, preprint.
6. R. A. Hunt, *On $L(p, q)$ spaces*, Enseignement Math. **12** (1966), 249–274.
7. W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. No. 217 (1979).
8. J. L. Krivine, *Sous espaces de dimension finie des espaces de Banach réticules*, Ann. of Math. (2) **140** (1976), 1–29.
9. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II: Function spaces*, Ergebnisse Math. Grenzgebiete, Bd. 97, Springer-Verlag, Berlin and New York, 1979.

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202

DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843