GEOMETRICAL PROPERTIES OF CONES AND THE COMPARISON OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

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ABSTRACT. We introduce here certain geometrical concepts related to the properties of cones in Banach spaces and show that these definitions are realizable. Then the results on the comparison of solutions are formulated where the ordering is introduced by means of a cone which is not required to be solid.

Introduction. The concept of quasimonotonicity, introduced in [1], has been extended and generalized in different ways. Whenever a partial ordering was introduced by means of a cone, either the solidity of the cone or strong restrictions on the relevant differential equations were required. The problem of the existence of differential equations in arbitrary closed sets is treated in [7]. However, when the set is a cone, the theorems of comparison (or invariance) found in [3 and 7], as well as in [8], involve either Lipschitz-like conditions, or the presence of interior points, even if questions of existence are treated separately from those of invariance. The distinction of the present approach from those publications known to the author lies in the domain of "boundary conditions" [7] which in [2, 4 and 8] are imposed on the whole cone if the cone is not solid. Here we investigate the possibility of comparing solutions for nonsolid cones and our "boundary conditions" are imposed only on a subset of the cone. Our goal is to introduce and illustrate the use of a certain geometrical concept. For that reason we select the simplest type of additional "boundary conditions" on the differential equation itself. They also have intrinsic geometrical significance in our situation, but for the solid cone case analogous conditions were used earlier in [3]. All terms used hereafter without any explanations have the same meanings as in [6].

1. Let B be a real Banach space and P a cone in B . An element $u \in P$ is called a *relatively-interior* one if a neighborhood R_u of the zero element in B exists such that for arbitrary $x \in R_u \cap P$ the element $(u - x) \in P$. A cone is called *presolid* if it contains a relatively-interior point. Let P be a presolid cone; then the set W of nonrelatively interior elements of P is called the *wall* of the cone.

REMARK. The above definition of relatively-interior element is equivalent to the following: The element $x \in P$ is relatively-interior iff $\forall \phi \in P, \exists \mu > 0$ such that $(x - \mu\phi) \in P$.

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PROOF. Given $\phi \in P$, $\exists n \in N$ such that $(x - n^{-1}\phi) \in P$, i.e., $\phi \in n(x - P)$. Thus

$$P = \bigcup_{n=1}^{\infty} [n(x - P) \cap P].$$

Notice that P and $n(x - P)$ are closed sets in the complete metric space B . Thus, by Baire's theorem, $\exists \phi_0 \in P$ and a real number $s > 0$ such that

$$(1) \quad \phi_0 + P_s \subset x - P,$$

where $P_s = \{h \in P: \|h\| < s\}$. Rewrite (1) as $x - \phi_0 - P_s \subset P$. Then we have

$$x - P_s = (x - \phi_0 - P_s) + \phi_0 \subset P,$$

as was required.

Let us show that there are Banach spaces with presolid cones having some other useful properties.

THEOREM. *There exists a real Banach space E with a cone K which satisfies the following conditions:*

- (a) *the linear hull of K is dense in the space E ;*
- (b) *the cone K is normal;*
- (c) *the cone K is not solid;*
- (d) *the cone K is minihedral;*
- (e) *an element $u \in K$ exists such that $x \in K$, $\|x\| \leq 1 \Rightarrow (u - x) \in K$.*

Thus the cone K is presolid.

PROOF. Let E be the set of all sequences of real numbers $x = (x_1, x_2, \dots, x_n, \dots)$, such that

$$\|x\| \equiv \sup\{|x_1|, |x_1 + x_2|, |x_2 + x_3|, \dots, |x_{n-1} + x_n|, \dots\}$$

is finite. With coordinatewise operations the set E is a real linear space and it is clear that $\|\cdot\|$ possesses the properties of a norm. It follows directly from the definition of the norm that convergence in this norm means coordinatewise convergence. Routine maneuvers then confirm that the space E is complete in this norm. Thus, E is a Banach space. Let K be the set of all elements $x \in E$, $x = \{x_k\}$ ($k = 1, 2, \dots$) such that for all k , $x_k \geq 0$ and $x_k \leq N_x$, where N_x is a positive number depending only on the element x . Let m be the normed space of bounded number sequences. The set K is a subset of the set m , $K \subset m$. On the other hand the set m is a part of the set E , $m \subset E$. Let us denote by $\|\cdot\|_m$ the norm in the space m . Then for all $x \in K$,

$$(2) \quad \|x\|_m \leq \|x\| \leq 2\|x\|_m.$$

The set K is closed in the space m and therefore it is closed in the space E also. It follows from this fact that K is a cone in the space E . Let us denote by X the closure of the set m in the norm $\|\cdot\|$ of the space E . X is a subspace of the space E . Thus, now the set K becomes a cone in the Banach space X . Let $x, y \in K$, $x = \{x_i\}$, $y = \{y_i\}$. Let us form the sequence $z = \{z_i\}$, $z_i = \max(x_i, y_i)$. Then $z \in X$ and $z = \sup(x, y)$. Therefore, part (d) of the theorem is proved. Let $x_1, x_2 \in K$, $\|x_1\| = \|x_2\| = 1$. Then from (2) it follows that $\|x_1 + x_2\| \geq \|x_1 + x_2\|_m > 1$, and thus part (b) is proved.

Let us take the element $u = (1, 1, \dots, 1, \dots) \in K$ and consider any element $x \in K$ with $\|x\| \leq 1$. Then it follows from (2) that $(u - x) \geq 0$. Thus, part (e) is proved. We will prove (c) by *reductio ad absurdum*. To this end let us suppose that K is a solid cone in E . Then K is a reproducing cone and the set X must coincide with the set m . From the definition of the norms $\|\cdot\|_m$ and $\|\cdot\|$ it then follows that $\|x\| \leq 2\|x\|_m \forall x \in X$.

Now we appeal to the following consequence of Banach's theorem on the inverse operator: If in a linear space two norms are given which convert this space into Banach spaces E_1 and E_2 , and in addition there is a positive number C_1 such that $\|x\|_2 \leq C_1\|x\|_1$ for all $x \in E$, then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. Accordingly, the norms $\|\cdot\|$ and $\|\cdot\|_m$ must be equivalent in the space m . Let us show that this is absurd. Let n be an arbitrary natural number and define the elements

$$x_n = (1, -2, 3, -4, 5, \dots, -2n, 2n-1, 2n-2, 2n-3, \dots, 1, 0, 0, \dots), \\ n = 1, 2, \dots$$

Then $x_n \in m$, $\|x_n\|_m = 2n$ and $\|x_n\| = 1$, for each n . This is a contradiction. Thus, part (c) of the theorem is proved.

Inasmuch as the convergence in E is coordinatewise, the linear hull of K is dense in E . That proves part (a), so the theorem is completely proved.

2. Let $f(x, t)$ be a continuous function defined on $B \times J$ with values in B , where B is a real Banach space and J is the interval $0 \leq t \leq T$ with $0 < T \leq \infty$. By the solution of the initial-value problem

$$dx/dt = f(x, t), \quad x(0) = x_0,$$

on $[0, a)$, $a \leq T$, we mean the function $x(t)$ having a strong derivative on that interval and satisfying the differential equation and the initial condition.

Let K be a presolid cone in space B and $L(K)$ the linear hull of K . Suppose the functions $f_1(x, t)$ and $f_2(x, t)$ have the properties mentioned above for $f(x, t)$, and the following conditions are satisfied:

$$(3) \quad F(p, x, t) \in L(K) \quad \forall p \in K, x \in B, t \in [0, a),$$

where

$$F(p, x, t) \equiv f_1(x + p, t) - f_2(x, t)$$

and

$$(4) \quad F(p, x, t) + Mp \in K_{\text{rip}} \quad \forall p \in W, x \in B, t \in [0, a).$$

Here K_{rip} is the set of relatively interior points of K , W is a wall of K and M is a positive real number.

Let us consider now two initial-value problems

$$dx_i/dt = f_i(x_i, t), \quad x_i(0) = x_{i0} \quad (i = 1, 2).$$

Suppose that $x_i(t)$ are solutions of these problems on the interval $[0, a)$.

THEOREM. Let conditions (3) and (4) be valid and let $x_{10} \geq x_{20}$. Then $x_1(t) \geq x_2(t)$ for $t \in [0, a)$.

PROOF. Let us denote by $p(t)$ the function

$$p(t) \equiv x_1(t) - x_2(t), \quad t \in [0, a),$$

and define $p_0 = x_{10} - x_{20}$.

The function $p(t)$ satisfies the differential equation $dp/dt = F(p, t)$, where $F(p, t) \equiv f_1[x_2(t) + p, t] - f_2[x_2(t), t]$, and the initial condition $p(0) = p_0 \in K$.

Let us denote by $\bar{t}$ the supremum of the set

$$\Omega = \{s: p(t) \geq 0 \text{ for all } t \in [0, s], 0 \leq s < a\}.$$

What must be proved is that $\bar{t} = a$.

Assume that $\bar{t} < a$. Then all the values $p(t)$ belong to the set K on the entire interval $[0, \bar{t}]$. Now we denote $p_{cr} = p(\bar{t})$ and use the formula

$$(5) \quad p(\bar{t} + \tau) = p_{cr} + \tau \cdot F(p_{cr}, \bar{t}) + \tau \cdot \beta(\tau)$$

from [5], where $\|\beta(\tau)\| \rightarrow 0$ as $\tau \rightarrow 0$ and $\beta(\tau) \in L(K)$ for small $\tau > 0$ by virtue of condition (3). The element p_{cr} cannot belong to K_{rip} , since otherwise for sufficiently small positive τ for all $t \in [\bar{t}, \bar{t} + \tau]$ we would have $p(t) \in K$, contrary to the definition of the number $\bar{t}$. Thus $p_{cr} \in W$. Now we make use of hypothesis (4) of the theorem. We have

$$p(\bar{t} + \tau) = (p_{cr} + \tau \cdot F(p_{cr}, \bar{t}) + \tau M p_{cr}) - \tau M p_{cr} + \tau \cdot \beta(\tau).$$

Here the element $(p_{cr} + \tau \cdot F(p_{cr}, \bar{t}) + \tau M p_{cr})$ is a relatively interior element of K .

Consequently, for small positive τ the elements $p(\bar{t} + \tau)$ belong to K . This violates the definition of $\bar{t}$ and therefore the theorem is proved.

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