SUBREPRESENTATIONS OF DIRECT INTEGRALS AND FINITE VOLUME HOMOGENEOUS SPACES

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ABSTRACT. We prove a result on representations of separable C^* -algebras which has application to, and was in fact motivated by, a problem concerning relations between unitary representations of a second countable locally compact group G and those of a closed subgroup K , when G/K is of finite volume. The result is that if an irreducible representation π is contained in $\int_X \pi_x d\mu(x)$, then $\pi \subseteq \pi_x$ for all x in a set of positive measure. With G and K as above, it follows that for each $\pi \in \hat{G}$ there exists $\sigma \in \hat{K}$ with $\pi \subseteq U^\sigma$, the induced representation. Frobenius reciprocity type results are derived as further consequences.

1. Introduction. In [1, §4] the question is raised of whether a group representation which is equivalent to a direct sum of irreducibles can also be equivalent to some nontrivial direct integral of other irreducibles. Theorem 2.1 below provides a negative answer to this question, and can be used to weaken the hypotheses of several theorems in [1, §4]. The proof is straightforward and elementary but seems to have been overlooked.

Let G be a second countable locally compact group and K a closed subgroup such that the quotient space G/K is of finite volume. An immediate consequence of Theorem 2.1 is that for each $\pi \in \hat{G}$, the space of unitary equivalence classes of irreducible representations of G , there exists $\sigma \in \hat{K}$ such that π is a subrepresentation of the induced representation U^σ . This result extends Theorem 2.3 of [4], where it is assumed that G/K is also compact, and provides additional evidence for the conjecture, known to hold when K is normal [4, Theorem 2.2], that if K is any closed subgroup of G such that G acts amenably on G/K , then each $\pi \in \hat{G}$ is weakly contained in U^σ for some $\sigma \in \hat{K}$. As further consequences of the above results, we obtain a necessary condition on $\pi \in \hat{G}$ in order that it satisfy Frobenius reciprocity with respect to all $\sigma \in \hat{K}$, and prove that this condition is also sufficient when K is normal in G .

Throughout this paper, all groups are assumed to be second countable and all Hilbert spaces and algebras separable. We use the same letter π for a unitary representation of a group G and the corresponding representation of the group C^* -algebra $C^*(G)$, and we write $\mathcal{H}(\pi)$ for the Hilbert space of π , π^p for the representation of a C^* -algebra determined by a positive functional p on the algebra, and $\pi \subseteq \sigma$ to mean that π is a subrepresentation of σ . The reader is referred to [3, §§3–4] for material on C^* -algebras associated with transformation groups.

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2. Containment and Frobenius reciprocity.

THEOREM 2.1. *Let X be a Borel space, μ a probability measure on X , and $x \rightarrow \pi_x$ a measurable field of representations of a separable C^* -algebra A . If π is an irreducible representation of A and $\pi \subseteq \int_X \pi_x d\mu(x)$, then π is unitarily equivalent to a subrepresentation of π_x for all x in a measurable set E of strictly positive measure.*

PROOF. Let y be a unit vector in $\mathcal{H}(\pi)$ given by a measurable vector field $x \rightarrow y_x$. The set $E \equiv \{x \in X: y_x \neq 0\}$ is a measurable set of strictly positive measure, and as y is cyclic for π it is clear that $\pi \subseteq \int_E \pi_x d\mu(x)$. We work only with E from now on and write μ for the restriction of μ to E . By replacing μ by an equivalent measure ν on E , which will be a probability measure, and whose Radon-Nikodym derivative $(d\nu/d\mu)(x) = \|y_x\|^2$, we may assume $\|y_x\| = 1 \forall x \in E$.

For $a \in A$, $x \in E$, let $p(a) = \langle \pi(a)y, y \rangle$, $p_x(a) = \langle \pi_x(a)y_x, y_x \rangle$. The space $\mathcal{S}(A)$ of positive functionals on A of norm ≤ 1 is a compact convex set in the weak*-topology, and if $\mathcal{S}(A)$ is given the Borel structure generated by this topology, it follows from the measurability of the functions $x \rightarrow p_x(a)$, $a \in A$, that the map $\Phi: E \rightarrow \mathcal{S}(A)$, given by $\Phi(x) = p_x$, is also measurable, so that for B Borel $\subseteq \mathcal{S}(A)$, $\Phi^{-1}(B)$ is measurable in E . For such B , $(\Phi_*\nu)(B) \equiv \nu(\Phi^{-1}(B))$ defines a regular probability measure on $\mathcal{S}(A)$. It follows that

$$p(a) = \int_E p_x(a) d\nu(x) = \int_{\mathcal{S}(A)} q(a) d(\Phi_*\nu)(q), \quad a \in A.$$

Hence, by Bauer's theorem [9, p. 8], $\Phi_*\nu$ is precisely the point mass at p , so $p_x = p$ a.e. on E . Thus $\pi \cong \pi^p \cong \pi^{p^*} \subseteq \pi_x$ a.e. on E , and we are done.

For a representation σ of a closed subgroup K of G , write $\langle U^\sigma, M^\sigma \rangle$ for the induced system of imprimitivity based on $(G, G/K)$, and L^σ for the corresponding representation of the crossed product algebra $C^*(G \times G/K)$ (see [3, §§3–4]). By the proof of Theorem 2.1, we have

PROPOSITION 2.2. *Let G be a second countable locally compact group and K a closed subgroup, with homogeneous space G/K of finite volume. Let $\pi \in \hat{G}$ and suppose $\pi|_K$ can be decomposed as a direct integral $\int \sigma d\nu(\sigma)$. Then $\pi \subseteq U^\sigma$ a.e. $(d\nu)$.*

PROOF. It is well known that $\pi \subseteq U^{\pi|_K}$ (see [8], for example), and that inducing preserves direct integrals, so that $U^{\pi|_K} = \int U^\sigma d\nu(\sigma)$. By Theorem 2.1, we need only choose a unit vector y in $\mathcal{H}(\pi) \subseteq \mathcal{H}(U^{\pi|_K}) = \int \mathcal{H}(U^\sigma) d\nu(\sigma)$ given by a measurable vector field $\sigma \rightarrow y_\sigma$ such that $y_\sigma \neq 0$ a.e. $(d\nu)$. However, any unit vector y in $\mathcal{H}(\pi)$ is cyclic for the representation $L^{\pi|_K}$ of $C^*(G \times G/K)$, or equivalently, for the algebra of operators generated by $U^{\pi|_K}$ and $M^{\pi|_K}$. This can be verified routinely and we omit the details. As $L^{\pi|_K} = \int L^\sigma d\nu(\sigma)$, it follows that $y_\sigma \neq 0$ a.e. $(d\nu)$.

When G/K is of finite volume, representations of G and of K partially satisfy Frobenius reciprocity relations. Let $d(\pi, \tau)$ denote the dimension of the space of intertwining operators between two representations π and τ . Then if π is a representation of G and σ is a representation of K , $d(\sigma, \pi|_K) \leq d(\pi, U^\sigma)$ ([8, §4] and [5, Corollary 4.8]), while a known sufficient condition that equality holds is that π be finite dimensional ([8, §4] and [6, Theorem 8.2]). Leaving aside questions of multiplicity and considering only irreducible representations $\pi \in \hat{G}$, $\sigma \in \hat{K}$, one has the

weaker results that (a) $\sigma \subseteq \pi|_K \Rightarrow \pi \subseteq U^\sigma$ and (b) if π is finite dimensional, then $\pi \subseteq U^\sigma \Rightarrow \sigma \subseteq \pi|_K$, also. Proposition 2.2 may be viewed as stating that $\pi \subseteq U^\sigma$ for "almost all" σ appearing (not necessarily as atoms) in a particular direct integral decomposition of $\pi|_K$, thus providing a partial generalization of implication (a) above.

Theorem 2.1 and Proposition 2.2 are used below to obtain a necessary condition on $\pi \in \hat{G}$ for π to satisfy implication (b) above, for all $\sigma \in \hat{K}$. This condition is weaker than finite dimensionality, and we prove it is also sufficient when K is normal.

PROPOSITION 2.3. *Let G/K be of finite volume, and let $\pi \in \hat{G}$. If $\pi \subseteq U^\sigma$ implies $\sigma \subseteq \pi|_K$, $\forall \sigma \in \hat{K}$, then $\pi|_K$ can be decomposed as a direct sum of elements of $\hat{K}$.*

PROOF. There exists a standard Borel space X with probability measure ν and a measurable field $x \rightarrow \sigma_x$ of irreducible representations of K such that $\pi|_K = \int_X \sigma_x d\nu(x)$ [2, Theorem 8.5.2]. By the hypothesis on π and Proposition 2.2, there exists a measurable subset E of X with $\nu(E) = 1$, such that $\sigma_y \subseteq \pi|_K = \int_X \sigma_x d\nu(x)$, $\forall y \in E$. For each $y \in E$, let $X_y = \{x \in X: \sigma_x \cong \sigma_y\}$. Then X_y is measurable, since singletons in $\hat{K}$ are Borel and the map of $X \rightarrow \hat{K}$, sending x into the unitary equivalence class of σ_x , is Borel [7, pp. 76–81]. By Theorem 2.1 and the fact that all the σ 's are irreducible, it follows that each X_y is of positive measure. Since X has finite measure and two sets of the form X_y are either identical or disjoint, it is clear that there are at most countably many such sets, say X_i , $1 \leq i \leq \infty$. Then $E \subseteq \bigcup_{i=1}^\infty X_i \subseteq X$ and $\pi|_K = \sum_{i=1}^\infty \int_{X_i} \sigma_x d\nu(x)$. As X is a standard Borel space, the restriction of ν to X_i is a standard measure and the result follows by Proposition 8.1.7 of [2].

PROPOSITION 2.4. *Let G be a second countable locally compact group, let K be a closed normal subgroup such that G/K is compact, and let $\pi \in \hat{G}$. Then $\pi \subseteq U^\sigma$ implies $\sigma \subseteq \pi|_K$, $\forall \sigma \in \hat{K}$, if and only if $\pi|_K$ can be decomposed as a direct sum of elements of $\hat{K}$. This in turn occurs if and only if $\pi|_K$ can be based on a finite G -orbit in $\hat{K}$.*

PROOF. Assume $\pi|_K$ can be decomposed as a direct sum of elements of $\hat{K}$, and let $\sigma \in \hat{K}$, $\sigma \subseteq \pi|_K$. Then $\pi \subseteq U^\sigma$ (§4 of [8] or Corollary 4.8 of [5]), and $\sigma \subseteq \pi|_K \subseteq U^\sigma|_K = \int_{G/K} x\sigma dx$ [6, Theorem 12.1], where dx is Haar measure on G/K and, for $x \in G$, $x\sigma$ is the unitary equivalence class of the representation of K given by $k \rightarrow \sigma(x^{-1}kx)$, $k \in K$. As $k\sigma \cong \sigma$ for all $k \in K$, the integral is indeed defined over G/K . Let $G_\sigma \equiv \{g \in G: g\sigma \cong \sigma\}$. A priori, G_σ need not even be closed in G unless K is type I. By Theorem 2.1 and the irreducibility of each $x\sigma$, however, G_σ/K is of positive Haar measure in G/K , and is thus open in G/K . Thus G_σ is open in G , G/G_σ is discrete, hence finite, and if $\{x_1, \dots, x_n\}$ is a set of coset representatives for G_σ in G ,

$$\pi|_K \subseteq U^\sigma|_K = \int_{G/K} x\sigma dx = l \cdot \sum_{i=1}^n \oplus x_i \sigma, \quad \text{where } l = [G_\sigma: K].$$

As $x(\pi|_K) \cong \pi|_K$ for all x in G , it follows that $\pi|_K$ is indeed based on the finite G -orbit $\{x_1\sigma, \dots, x_n\sigma\}$ in $\hat{K}$. Finally, if $\pi \subseteq U^\tau$ for some $\tau \in \hat{K}$, we have $x_i\sigma \subseteq \pi|_K \subseteq U^\tau|_K = \int_{G/K} x\tau dx$, so as above, by Theorem 2.1, the G -orbit of σ equals the G -orbit of τ , and for some j , $\tau \cong x_j\sigma \subseteq \pi|_K$.

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