ON THE PRODUCT OF CLASS A SEMIGROUPS OF LINEAR OPERATORS

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ABSTRACT. The present paper extends a result of Trotter concerning the product of c_0 semigroups. We show that the product of two commuting semigroups of class A is again a semigroup of class A and that its generator is the sum (or its closure) of the generators of the factor semigroups. This provides a partial answer in the affirmative to a question of Hille and Phillips concerning the sum of unbounded operators [3, p. 417]. Sufficient conditions are also given in terms of the generators and their domains so that the semigroups commute.

1. Introduction. Let T_t and T'_t be one parameter semigroups of bounded linear operators on a Banach space. Trotter [4] showed that if T_t, T'_t are of class c_0 with respective generators A, A' and if T_t and T'_s commute for all positive t and s , then the product $S_a(t) = T_t T'_{at}$, $a > 0$, is again a c_0 semigroup whose generator is the closure of $A + aA'$. This result was further generalized by dropping the assumption of commutativity [1, 2, 4].

The purpose of the present paper is to consider the product in a more general class of semigroups, namely those of class A . Assuming commutativity we extend the above result of Trotter. This also provides an answer in the affirmative to a special case of the question raised in [3, p. 417]. It is well known that the answer to this question in the general case is in the negative [2, 4]. In Theorem 3 we give sufficient (but not necessary) conditions for commutativity of the semigroups in terms of the generators and their domains.

2. Preliminaries. Let X be a complex Banach space and $L(X)$ the Banach algebra of all bounded linear operators on X . For a linear operator A from X to itself we let $D(A)$ denote the domain of A , $\rho(A)$ the resolvent set of A and $R(\lambda; A)$ the resolvent operator of A . If A is closed, then $R(\lambda; A) \in L(X)$ for every λ in $\rho(A)$. A semigroup of bounded linear operators on X is a mapping from the positive real numbers to the space $L(X)$ that satisfies the semigroup property: $T_{t+s} = T_t T_s$, $t, s > 0$.

In what follows we assume that the semigroups are strongly continuous for $t > 0$. The infinitesimal operator of T_t is defined by

$$(1) \quad A_0 x = s\text{-}\lim_{h \rightarrow 0+} h^{-1}(T_h x - x).$$

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A_0 is a linear operator from X to itself, and its domain consists of all those elements x in X for which the limit exists in the strong sense. The range of T_t , denoted by X_0 , is the subspace

$$X_0 = \bigcup \{T_t X : t > 0\}.$$

The type of T_t is denoted by ω_0 , where $\omega_0 = \lim_{t \rightarrow \infty} t^{-1} \log \|T_t\|$. T_t is a c_0 semigroup if it is strongly continuous at 0. In this case A_0 is closed and its domain is dense in X .

In the absence of continuity at the origin, semigroups of class A may be introduced as follows [3]:

DEFINITION. A strongly continuous semigroup T_t , $t > 0$, is said to be of class A if the following conditions are satisfied:

- (i) X_0 is dense in X .
- (ii) There exists $\omega_1 > \omega_0$ such that for each λ , $\operatorname{Re} \lambda > \omega_1$, there is an operator $J(\lambda) \in L(X)$ with the following properties:

(a)

$$J(\lambda)x = \int_0^\infty e^{-\lambda t} T_t x \, dt \quad \text{for } x \in X_0.$$

(b) $\|J(\lambda)\|$ is bounded in the half-plane $\operatorname{Re} \lambda > \omega_1$.

(c) $\lim_{\lambda \rightarrow \infty} \lambda J(\lambda)x = x$ for every x in X .

For a semigroup of class A the infinitesimal operator A_0 is not closed; however it has a smallest closed extension A and $J(\lambda) = R(\lambda; A)$ for $\operatorname{Re} \lambda > \omega_1$ [3, p. 344]. The operator A is called the infinitesimal generator of T_t , and T_t is related to the resolvent of A through

$$(2) \quad T_t x = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} R(\lambda; A)x \, d\lambda, \quad \operatorname{Re} \lambda > \omega_1 \text{ and } x \in D(A^2).$$

Finally to simplify formulas, T_t may be replaced by an equivalent semigroup for which $\omega_0 < \omega_1 < 0$. This we assume throughout the paper.

3. The product semigroup. The connection between the semigroup of operators $\{T_t : t > 0\}$ and the resolvent operators $\{R(\lambda; A) : \operatorname{Re} \lambda > \omega_1\}$ leads one to expect that commutativity ought to be shared by both families. The following theorem shows that this is indeed the case.

THEOREM 1. *Let T_t and T'_t be two semigroups of class A on a Banach space. Then the semigroups commute ($T_t T'_s = T'_s T_t$ for positive t, s) if and only if the resolvents $R(\lambda; A)$ and $R(\mu; A')$ commute for all sufficiently large λ and μ .*

This theorem is an easy consequence of

LEMMA 1. *Let T_t be a semigroup of class A and let S be a bounded linear operator on the Banach space X . Then S commutes with all T_t if and only if S commutes with $R(\lambda; A)$ for all large λ .*

PROOF. First suppose that $ST_t = T_t S$ for all positive t . Suppose $x \in X_0$. Then $x = T_s y$ for some y in X and $s > 0$, and thus $Sx = ST_s y = T_s S y$, which is also in X_0 .

Hence by formula (ii)(a) with x replaced by Sx ,

$$\begin{aligned} R(\lambda; A)Sx &= \int_0^\infty e^{-\lambda t} T_t Sx \, dt = \int_0^\infty e^{-\lambda t} S T_t x \, dt \\ &= S \int_0^\infty e^{-\lambda t} T_t x \, dt = SR(\lambda; A)x. \end{aligned}$$

As X_0 is dense and the operators are bounded, we get $R(\lambda; A)S = SR(\lambda; A)$.

Conversely, suppose $R(\lambda; A)S = SR(\lambda; A)$ for all sufficiently large real λ . By analyticity in λ , this relation holds also in the half-plane $\operatorname{Re} \lambda > \omega_1$. Also $SR^2(\lambda; A) = R^2(\lambda; A)S$, so S maps $D(A^2)$ into itself. Therefore, by virtue of the Laplace inversion formula (2) for T_t on $D(A^2)$, we get $T_t Sx = S T_t x$ for all x in $D(A^2)$. Since the latter domain is dense, it follows that S and T_t commute.

It is clear that the product $S(t) = T_t T'_t$, $t > 0$, is a strongly continuous semigroup. As noted earlier, this is not continuous at 0 unless the factor semigroups are of class c_0 . It turns out that in the case of semigroups of class A (Lemma 2), the continuity set of $S(t)$, that is, the set of all x in X for which $S(t)x \rightarrow x$ as $t \rightarrow 0$, is dense in X .

REMARK. Under the hypotheses of Theorem 1, we have

(1) $A'R(\lambda; A) = R(\lambda; A)A'$ on $D(A')$, in particular $R(\lambda; A)$ maps $D(A')$ into itself. Similarly with A' and A interchanged.

(2) Let $K_\lambda^n = R^n(\lambda; A)R^n(\lambda; A')$, n a positive integer. Then the range space $R(K_\lambda^n)$ is dense in X for each n .

(1) follows readily by Lemma 1, formula (1) and the fact that A is the closure of A_0 .

Similarly, (2) can be easily verified using the facts that $D(A)$ and $D(A')$ are dense, and $R(\lambda; A)$ and $R(\lambda; A')$ commute.

LEMMA 2. If T_t and T'_t are commuting semigroups of class A , equivalently if their resolvents commute, and if $S(t) = T_t T'_t$, then $S(t)x \rightarrow x$ as $t \rightarrow 0$ for all x in $R(K_\lambda^2)$.

PROOF. From the theory of class A semigroups presented in Hille and Phillips, we know that $T_t R^2(\lambda; A) \rightarrow R^2(\lambda; A)$ strongly as $t \rightarrow 0$ (since $T_t x \rightarrow x$ for all x in $D(A^2)$). Hence

$$S(t)K_\lambda^2 = T_t R^2(\lambda; A) \cdot T'_t R^2(\lambda; A') \rightarrow K_\lambda^2 \quad \text{strongly.}$$

In particular, this shows that the "continuity set" of $S(t)$ is dense. Moreover, the integral

$$(3) \quad J(\lambda) = \int_0^\infty e^{-\lambda t} S(t)x \, dt$$

exists whenever $x \in R(K_\lambda^2)$ and $\operatorname{Re} \lambda > 0$.

We now state and prove the main result.

THEOREM 2. If T_t and T'_t are commuting semigroups of class A , then the product $S(t) = T_t T'_t$ is also a semigroup of class A , whose generator is the closure of $A + A'$.

PROOF. First we show that $S(t)$ is of class A . We note that (i) follows directly from Lemma 2 and Remark 2. To verify (ii) we begin by considering the operator defined by

$$(4) \quad L(\lambda, \sigma)x = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} R(\mu; A')R(\lambda - \mu; A)A'^2K_\sigma^2x \frac{d\mu}{\mu^2}$$

where $\max(w, w') < \gamma < 0$ (w and w' being the types of T_t and T'_t respectively), $\operatorname{Re} \lambda > 0$ and $\sigma > 0$. To see that the right side of (4) is well defined for each x in X , we note that since A' commutes with $R(\sigma; A')$ on $D(A')$,

$$A'^2K_\sigma^2 = [A'R(\sigma; A')]^2R^2(\sigma; A).$$

Since both $A'R(\sigma; A')$ and $R(\sigma; A)$ are bounded operators on X so is $A'^2K_\sigma^2$. Moreover it follows from (ii)(b), (c) (or from [3, Theorem 12.5.1]) that $A'^2K_\sigma^2$ and $\sigma^2A'^2K_\sigma^2$ are bounded in norm for all sufficiently large real σ , while $R(\mu; A')$ and $R(\lambda - \mu; A)$ are norm bounded on the line γ . Therefore the integral in (4) is absolutely convergent for each x in X and defines a bounded operator on X . Moreover there is a constant M such that

$$(5) \quad \|\sigma^2L(\lambda, \sigma)x\| \leq M \cdot \|x\| \quad \text{for all } \operatorname{Re} \lambda, \sigma > 0.$$

Next, we note from Remark 1 that A'^2 and $R(\lambda - \mu; A)$ commute on $R(K_\sigma^2)$, hence making use of (ii)(a) and (2) we have that

$$\begin{aligned} L(\lambda, \sigma)x &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} A'^2R(\mu; A') \left[\int_0^\infty \bar{e}^{(\lambda-\mu)t} T(t) K_\sigma^2x \, dt \right] \frac{d\mu}{\mu^2} \\ &= \int_0^\infty \bar{e}^{\lambda t} T(t) \left[\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\mu t} A'^2R(\mu; A') K_\sigma^2x \frac{d\mu}{\mu^2} \right] dt \\ &= \int_0^\infty \bar{e}^{\lambda t} S(t) K_\sigma^2x \, dt = J(\lambda) K_\sigma^2x. \end{aligned}$$

Now $J(\lambda)$ and K_σ^2 commute on $R(K_\sigma^2)$, hence if $x \in R(K_\sigma^2)$,

$$\sigma^2L(\lambda, \sigma)x = \sigma^2J(\lambda)K_\sigma^2x = \sigma^2K_\sigma^2J(\lambda)x \rightarrow J(\lambda)x \quad \text{as } \sigma \rightarrow \infty.$$

It follows from this and (5) that

$$(6) \quad \|J(\lambda)x\| \leq M \cdot \|x\| \quad \text{for all } \operatorname{Re} \lambda > 0, x \in R(K_\sigma^2).$$

Since $R(K_\sigma^2)$ is dense in X , $J(\lambda)$ has a unique bounded extension to all of X which we again denote by $J(\lambda)$. To complete the proof of (ii)(a) let $x \in X_0$ (the range of $S(t)$); then there exists y in X and $s > 0$ such that $x = S(s)y$. Let (y_n) be a sequence in $R(K_\lambda^2)$ which converges to y . Since $S(t)$ commutes with K_λ^2 , $S(t)y_n$ is in $R(K_\lambda^2)$ for all n ; therefore by (3),

$$J(\lambda)S(s)y_n = \int_0^\infty \bar{e}^{\lambda t} S(t+s)y_n \, dt.$$

As $\|S(t+s)\|$ is bounded for all $t \geq 0$ we can take the limit as $n \rightarrow \infty$, thus obtaining (ii)(a). It is also clear that (ii)(b) follows from (6).

To prove (ii)(c) we note first that since

$$R(\lambda - \mu; A)x = (\lambda - \mu)^{-1}(I + R(\lambda - \mu; A)A)x$$

on $D(A)$, the integrand in the right side of (4) may be written in the form

$$\frac{1}{\mu^2(\lambda - \mu)} \cdot R(\mu; A') [A'^2 K_\sigma^2 + R(\lambda - \mu; A) A A'^2 K_\sigma^2] x,$$

and once more Remark 1 implies

$$A A'^2 K_\sigma^2 = A R(\sigma; A) [A' R(\sigma; A')]^2 R(\sigma; A),$$

which is bounded in norm for sufficiently large σ . Therefore the integrand in (4) for $\lambda L(\lambda; \sigma)x$, $\lambda > 0$, is dominated in norm by $K \cdot \|x\| \cdot \lambda / (|\mu^2| \cdot |\lambda - \mu|)$, hence $\|\lambda L(\lambda, \sigma)x\| \leq \beta \cdot \|x\|$, all $\lambda > 0$. If we apply the argument used in the proof of (6) we obtain

$$(7) \quad \|\lambda J(\lambda)\| \leq c \quad \text{for all positive } \lambda.$$

Next, since T_t and T'_t are of negative type, $S(t)$ is also of negative type. Hence $\|S(t)\|$ is bounded in each interval $t \geq k$, $k > 0$. Furthermore, for each x in the continuity set C of $S(t)$, $\|S(t)x\|$ is uniformly continuous in each interval $0 \leq t \leq k$, and therefore for all t , since $\|S(t)\|$ is bounded for $t \geq k$. To finish the proof of (ii)(c), let $x \in C$ and choose $\delta > 0$ (δ depends on x) such that for all t ,

$$\|S(t+s)x - S(t)x\| < \varepsilon \quad \text{whenever } 0 < s < \delta,$$

but $\|x\| \leq \sup_{t>0} \|S(t)x\|$, $x \in C$, hence

$$\|\lambda J(\lambda)x - x\| \leq \sup_{t>0} \lambda \left\| \int_0^\infty e^{-\lambda s} \{S(t+s)x - S(t)x\} ds \right\|.$$

Splitting the interval of integration into $(0, \delta)$ and (δ, ∞) and taking the limit as $\lambda \rightarrow \infty$, we see that $\lambda J(\lambda)x \rightarrow x$. Since C is dense, it follows in view of (7) and the Banach-Steinhaus theorem that this limit actually holds for all x in X .

Second, we show that the generator of $S(t)$ is the closure of $A + A'$. Let Ω_0 denote the infinitesimal operator of $S(t)$; then Ω_0 is an extension of $A_0 + A'_0$ on $D(A_0) \cap D(A'_0)$. To see this we write

$$\Omega_t x = t^{-1}(S(t)x - x) = t^{-1}(T_t x - x) + t^{-1}(T'_t - I)T_t x,$$

and note that if $x \in D(A_0) \cap D(A'_0)$, the right side tends to $A_0 x + A'_0 x$ as $t \rightarrow 0$. Now, since we have shown that $S(t)$ is of class A , Ω_0 has a smallest closed extension Ω , which is by definition the infinitesimal generator of $S(t)$. We first show that Ω is an extension of $A + A'$ on $R(K_\lambda)$, that is to say, $R(K_\lambda) \subset D(\Omega)$ and

$$(8) \quad \Omega x = Ax + A'x \quad \text{for } x \in R(K_\lambda).$$

Let $x \in R(K_\lambda)$. Then $x = K_\lambda y$ for some y in X . Let (y_n) be a sequence from $R(K_\lambda^2)$ which converges to y . Then $K_\lambda y_n \rightarrow x$ as $n \rightarrow \infty$. But $K_\lambda y_n$ belongs to $D(A^3) \cap D(A'^3)$, which is contained in $D(A_0) \cap D(A'_0)$ according to [3, p. 345], hence

$$\Omega_0 K_\lambda y_n = A_0 K_\lambda y_n + A'_0 K_\lambda y_n \quad \text{for all } n.$$

Since A and A' are closed and $R(K_\lambda) \subset D(A) \cap D(A')$, AK_λ and $A'K_\lambda$ are bounded operators on X . Therefore the right side tends to a limit as $n \rightarrow \infty$, namely $Ax + A'x$, hence the left side has a limit, which is by definition Ωx .

Finally, to show that Ω is the closure of $A + A'$ on $D(A) \cap D(A')$, let $x \in D(A) \cap D(A')$. It is clear that $R(\lambda; A)x \in R(K_\lambda)$ and, in view of (5) and Remark 1, we have

$$\Omega(\lambda R(\lambda; A))x = (A + A')(\lambda R(\lambda; A))x = \lambda R(\lambda; A)(A + A')x.$$

As $\lambda \rightarrow \infty$, $\lambda R(\lambda; A)x \rightarrow x$ and $\lambda R(\lambda; A)(A + A')x \rightarrow (A + A')x$. This shows that Ω is an extension of $A + A'$. That Ω is the smallest closed extension may be proved as in [4, Theorem 1].

It is possible to give sufficient conditions in terms of the generators and their domains so that the semigroups commute, hence the conclusion of Theorem 2 holds.

THEOREM 3. *Let T_t and T'_t be semigroups of class A whose generators A and A' satisfy:*

- (a) $D(A) \subseteq D(A')$,
- (b) $D(A'A) \subseteq D(AA')$,
- (c) $AA'z = A'Az$, $z \in D(A'A)$.

Then $S(t) = T_t T'_t$ is of class A and its generator is the closure of $A + A'$.

PROOF. According to Theorem 1, it suffices to show that the resolvent operators commute for all sufficiently large λ and μ . Let $x \in X$ and put

$$z = R(\lambda; A)R(\mu; A')x.$$

Then $z \in D(A)$ and $(\lambda I - A)z \in D(A')$ and, by (a), $z \in D(A')$, hence $Az \in D(A')$, that is, $z \in D(A'A)$. Making use of (c) we obtain

$$x = (\mu I - A')(\lambda I - A)z = (\lambda I - A)(\mu I - A')z.$$

Thus $z = R(\mu; A')R(\lambda; A)x$.

It is worth noting that the above conditions are not necessary, even in the case of c_0 semigroups, as examples indicate [2, 3, p. 333 and 4].

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