A NOTE CONCERNING FOX'S PAPER ON FENCHEL'S CONJECTURE

T. C. CHAU

ABSTRACT. The object of this note is to point out and correct an error in the paper [2] of Fox, purporting to prove Fenchel's conjecture that a finitely generated, infinite Fuchsian group has a torsion-free normal subgroup of finite index. Fox divided the proof of the main result in his paper into four cases and an error occurs in the proofs of Cases (III) and (IV). A direct proof of Case (III) was given. While a direct proof of Case (IV) can be given, the author shows that it also follows indirectly from the result in the paper [1] of Burns and Solitar.

Given any three integers $a > 1$, $b > 1$ and $c > 1$, there exist permutations A of order a and B of order b , such that AB has order c .

Fox divided the proof of this lemma into the four cases:

(I) $ab \equiv 0 \pmod{2}$ and $\max\{a, b\} \leq c < a + b$,

(II) $ab \equiv 0 \pmod{2}$ and $c \geq a + b$,

(III) $a \equiv b \equiv c \equiv 1 \pmod{2}$,

(IV) $a \equiv b \equiv 1$ and $c \equiv 0 \pmod{2}$,

and subcases. In the subcase $m(a + b - 2) + 2 \leq c \leq (m + 1)(b + 1) - 1$ of Case (III) where m is the largest integer not exceeding $(c - 1)/(b + 1)$, and it is assumed that $a \leq b < c$, the number of distinct symbols in the permutations [2, p. 64]

$$B = (v_1^1 \cdots v_s^1 w_{b-s-1}^1 \cdots w_1^1 p_2) \cdot \prod_{i=2}^m (v_1^i w_{b-2}^i \cdots w_1^i p_{i+1}),$$

$$A = (p_1 u_1^1 \cdots u_{a-s-1}^1 v_1^1 \cdots v_s^1) \cdot \prod_{i=2}^m (p_i u_1^i \cdots u_{a-2}^i v_1^i) \cdot A',$$

where

$$A' = (p_{m+1} p_2^0 \cdots p_a^0) \cdot \left(\prod_{i=m-q+1}^m A_1^i \cdots A_{b-2}^i \right) \cdot A_1^{m-q} \cdots A_r^{m-q},$$

and

$$A_j^i = (w_j^i p_{j2}^i \cdots p_{ja}^i),$$

is $c - 2s$, not c as Fox claims, and the product AB is a cycle of length $c - 2s$, not c . Furthermore, Fox's claim that B has two consecutive symbols which do not occur in A for $a \leq b < c$ is false; whereas this is so when $a < b < c$, it is no longer the case

Received by the editors September 20, 1982.

1980 *Mathematics Subject Classification*. Primary 20F05; Secondary 20B27.

©1983 American Mathematical Society
0002-9939/82/0000-1055/\$01.50

when $a = b < c$. (Although this property of A and B is not required for the proof of Case (III), it is used by Fox in the proof of Case (IV).)

We now show how Fox's proof of this subcase of Case (III) and also Case (IV) can be corrected. Recall that the subcase in question is that where $m(a + b - 2) + 2 \leq c \leq (m + 1)(b + 1) - 1$, and where $a \leq b < c$ and $a \equiv b \equiv c \equiv 1 \pmod{2}$. We consider first the case where $a < b$ or $m > 1$. Divide $c - m(a + b - 2) - 2$ by $a - 1$, obtaining

$$c - m(a + b - 2) - 2 = t(a - 1) - s, \quad 0 \leq s < a - 1.$$

Since $s \equiv c \equiv 1 \pmod{2}$, we have $1 \leq s \leq a - 2$ and then $t > 0$. Since $3 \leq a \leq b$, we must have $a + b - 5 \geq 0$. Hence $a + b - 5 \leq m(a + b - 5)$, so that $b - ma + 3m - 3 \leq m(b - 2) - (a - 2) < \{m(b - 2) - a + 2\}(a - 1)$, since $m(b - 2) - a + 2 > 0$ in view of the assumption that either $a < b$ or $m > 1$. Thus

$$\begin{aligned} t(a - 1) &= c - m(a + b - 2) - 2 + s \\ &\leq (m + 1)(b + 1) - 1 - m(a + b - 2) - 2 + a - 2 \\ &= b - ma + 3m - 3 + a - 1 \\ &< \{m(b - 2) - a + 2\}(a - 1) + (a - 1) \\ &= \{m(b - 2) - a + 3\}(a - 1), \end{aligned}$$

and therefore $0 < t < m(b - 2) - a + 3$, whence $0 < t \leq m(b - 2) - a + 2$. Divide $t - 1$ by $b - 2$ obtaining

$$t - 1 = q(b - 2) + r, \quad 0 \leq r < b - 2.$$

Since $t - 1 \geq 0$, we must have $q \geq 0$. Since

$$t - 1 \leq m(b - 2) - a + 1 = (m - 1)(b - 2) + b - a - 1,$$

it follows that $q \leq m - 1$ and that $r \leq b - a - 1$ if $q = m - 1$. It can now be verified directly that the number of distinct symbols in A and B is c and the product AB is a cycle of order c . This proves Case (III) in the case $a < b$ or $m > 1$.

Note (for use in Case (IV)) that the symbols p_1, u_1^1 occur consecutively in A but do not occur in B , while assuming $b > a$, the symbols w_{b-a}^1, w_{b-a+1}^1 occur consecutively in B but do not occur in A . Hence in this situation Case (IV) follows as in [2].

To finally dispose of Case (III), we still need to consider the case $a = b$ and $m = 1$. Arguing as above we obtain $0 < t \leq m(b - 2) - a + 3$ (i.e. equality between t and $m(b - 2) - a + 3$ is now possible), i.e. $0 < t \leq b - a + 1$ whence $t = 1$. Hence $c = 3a - s - 1$ and it can now be verified that the permutations

$$\begin{aligned} B &= (v_1^1 \cdots v_s^1 w_{b-s-1}^1 \cdots w_1^1 p_2), \\ A &= (p_1 u_1^1 \cdots u_{a-s-1}^1 v_1^1 \cdots v_s^1) (p_2 p_2^0 \cdots p_a^0) \end{aligned}$$

have the required property; in fact AB is a cycle on the $c = 3a - s - 1$ symbols involved in A and B . This concludes the proof of Case (III). However, here it is not the case that B has two consecutive symbols which are not in A , so that in this situation Case (IV) can no longer be established by the argument given by Fox in [2]. Although a direct verification of these remaining subcases of Case (IV) can be given, they (in fact Case (IV) in general) also follow indirectly from the result in [1]. In [1]

it is shown using Cases (II) and (III) above (and the additional fact that in each of these cases the total number of symbols permuted is exactly c) that in particular the group $G = \langle x, y \mid x^a = y^b = 1, (xy)^c = 1 \rangle$ where $a \equiv b \equiv 1 \pmod{2}$, $c \equiv 0 \pmod{2}$ has a torsion-free subgroup H say, of index $n = 2 \times \text{lcm}(a, b, c)$. It follows easily from the fact that the elements x, y, xy actually have the orders a, b, c respectively in G (see [1]), that the respective images of x, y and xy under the homomorphism

$$\varphi: g \mapsto (Hg_1g, \dots, Hg_ng), \quad g \in G,$$

are degree- n permutations A, B, C of orders a, b, c such that $AB = C$.

The author thanks R. G. Burns for suggesting a problem leading to the present work and for checking the details, and D. Solitar for a helpful remark.

REFERENCES

1. R. G. Burns and D. Solitar, *Indices of torsion-free subgroups of Fuchsian groups*, Proc. Amer. Math. Soc. (to appear).
2. R. H. Fox, *On Fenchel's conjecture about F-groups*, Math. Tidsskr. B (1952), 61–65.

DEPARTMENT OF MATHEMATICS, LAURENTIAN UNIVERSITY, SUDBURY, ONTARIO, CANADA PE3 2C6