AN ISOMORPHISM THEOREM FOR VALUATED VECTOR SPACES

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ABSTRACT. The following isomorphism theorem is proved for valuated vector spaces. Let N and N' be nice subspaces of free valuated vector spaces F and F' , respectively. If N and N' have isomorphic basic subspaces and if the quotient spaces F/N and F'/N' are isomorphic, there exists an isomorphism from F onto F' that maps N onto N' and induces the given isomorphism on the quotient spaces. In particular, N and N' are isomorphic.

Throughout let Γ denote a fixed totally ordered set with suprema; every nonempty subset of Γ has a least upper bound in Γ . Adjoin to Γ an element ∞ that is considered larger than every element of Γ . Following [2] and [4], we define a valuated vector space to be a vector space V with values in $\{\Gamma, \infty\}$ satisfying the following conditions:

- (1) $|x + y| \geq \min\{|x|, |y|\}$ for all $x, y \in V$.
- (2) $|cx| = |x|$ if c is a nonzero scalar.
- (3) $|x| = \infty$ if and only if $x = 0$.

We shall refer to a valuated vector space simply as a valuated space. If V and W are valuated spaces, a map from V to W is a linear transformation that does not decrease values. Products and coproducts exist [2], and the coproduct of valuated spaces $\{V_i\}_{i \in I}$ is nothing more than the direct sum $\sum_{i \in I} \oplus V_i$ where the value of $x = \sum x_i$ with $x_i \in V_i$ is given by $|x| = \min\{|x_i|\}$.

The structure of valuated vector spaces is much more complicated than the structure of (nonvaluated) vector spaces. A fundamental feature of vector spaces that is missing in valuated spaces is the existence of a basis for every space. Naturally, a valuated space V has a basis $\{x_i\}$ when V is the coproduct of the one-dimensional spaces $\langle x_i \rangle$; in such case, V is said to be free. Thus having a basis and being free are equivalent. Every countably generated valuated space has a basis [1]. But in the uncountable case, even a subspace of a space with a basis need not have a basis; the author gave a necessary and sufficient condition for a subspace of a free valuated space to be again free in [3]. Although not every valuated space has a basis, it is true that every valuated space V has a *basic* subspace B . By this we mean that B has a basis and, for each nonzero $x \in V$, there exists $b \in B$ such that $|x - b| > |x|$. The latter property, in some sense, makes B dense in V ; see Theorem 4 in [2].

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If W is a subspace of the valued space V , the valuation on the quotient space is given by

$$|x + W| = \sup\{|x + w| : w \in W\},$$

and the subspace W is said to be a *nice* subspace of V if for every x the supremum is obtained, that is, if $|x + W| = |x + w|$ for some $w \in W$. It was once thought that if V had a basis then every nice subspace W of V also would have a basis, but it turned out differently [5] (see also Theorem 6.3 in [4]). Following [3] and [4], we call a nice subspace of a free space an NSF-space. Equivalently, an NSF-space is a nice subspace of a valued vector space that has a basis. Although NSF-spaces are not always free, we are able to prove here a structural uniqueness theorem for NSF-spaces. Our main result is the following isomorphism theorem.

THEOREM. *Let N and N' be nice subspaces of free valued vector spaces F and F' , respectively. If N and N' have isomorphic basic subspaces and if the quotient spaces F/N and F'/N' are isomorphic (all in the category of valued spaces), there exists an isomorphism from F onto F' that takes N onto N' and induces the given isomorphism on the quotient spaces. In particular, N and N' are isomorphic.*

PROOF. Let $\sigma: B \rightarrow B'$ be an isomorphism between basic subspaces of N and N' , respectively, and let $\phi: F/N \rightarrow F'/N'$ be an isomorphism between the quotient spaces. First, observe that if C/N is a basic subspace of F/N then $C = N \oplus A$ is the coproduct of N and a free subspace A of F . We use $\oplus$ throughout for coproduct. Moreover, $B \oplus A$ is a basic subspace of F since B is basic in N and $C/N \cong A$ is basic in F/N . Likewise, $B' \oplus A'$ is basic in F' if A' is chosen similarly to A . But $A \cong C/N$ and $A' \cong C'/N'$ are isomorphic because they are basic subspaces of isomorphic spaces (Theorem 4 in [2]). By hypothesis, $B \cong B'$. Therefore $A \oplus B \cong A' \oplus B'$, which implies that

$$F \cong A \oplus B \cong A' \oplus B' \cong F'$$

since F and F' are basic subspaces of themselves. Thus we can write $F = \sum_{i \in I} \langle x_i \rangle$ and $F' = \sum_{i \in I} \langle y_i \rangle$, where $|x_i| = |y_i|$ for each i . If $J \subseteq I$, define $F_J = \sum_{j \in J} \langle x_j \rangle$ and $F'_J = \sum_{j \in J} \langle y_j \rangle$.

Suppose that for some subset J of I that M and M' are subspaces of F and F' containing F_J and F'_J , respectively, such that M/F_J and M'/F'_J are finitely generated. Further, assume that $\pi: M \rightarrow M'$ is an isomorphism satisfying the following conditions:

- (1) π maps F_J onto F'_J .
- (2) $\pi(a) + N' = \phi(a + N)$ if $a \in M$.
- (3) $|b + a| > |b| \Leftrightarrow |\sigma(b) + \pi(a)| > |\sigma(b)|$ whenever $b \in B$ and $a \in M$.

Observe that if $M = F$ then $M' = F'$ since $M'/F'_J \cong M/F_J$ and M is finitely generated over F , and in this case the conclusion of the theorem follows from condition (2). Thus our aim is to enlarge the domain of π (until we obtain $M = F$). Suppose that $x \in F$ but $x \notin M$. Since M/F_J is finitely generated and since F_J is a direct summand (and consequently a nice subspace) of F , M must be a nice subspace

of F ; see Lemma 6 in [2]. Likewise, M' is a nice subspace of F' . It follows that $\langle x, M \rangle = \langle x + m \rangle \oplus M$ for some $m \in M$, and, by replacing x by $x + m$, we may assume that $\langle x, M \rangle = \langle x \rangle \oplus M$. We seek $y \in F'$ such that $|x| = |y|$, $\langle y, M' \rangle = \langle y \rangle \oplus M'$ and $\phi(x + N) = y + N'$. Two cases need to be distinguished. Even though N is nice, the first case cannot be eliminated because x has already been chosen orthogonal to M .

Case 1. $|x + N| > |x|$. Let $\phi(x + N) = w + N'$ where the representative w is chosen so that $|w| > |x|$. Choose $n \in N$ so that $|x + n| > |x|$, and choose $b \in B$ so that $|n - b| > |n| = |x|$. Observe that $|x + b| > |x| = |b|$. Define $b' = \sigma(b)$ and set $y = w - b'$. Since $|w| > |x|$ and $|b'| = |b| = |x|$, we know that $|y| = |x|$, and $\phi(x + N) = w + N' = y + N'$ since $b' \in N'$. In order to show that $\langle y, M' \rangle = \langle y \rangle \oplus M'$, assume the contrary and suppose that $|y + m'| > |y|$ for some $m' \in M'$. This immediately yields $|m' - b'| > |y| = |b'|$. Condition (3) implies that $|m - b| > |b|$, where $\pi(m) = m'$. But this leads to $|x + m| > |x|$ because $|x + b| > |x| = |b|$. However, $|x + m| > |x|$ is impossible because x is orthogonal to M . Thus we conclude that $\langle y, M' \rangle = \langle y \rangle \oplus M'$. The mapping π is extended by mapping x onto y . Furthermore, condition (1) is automatically retained by the extension, and we have chosen y so that condition (2) is retained. We need to verify that condition (3) survives. Toward that end, let c denote an arbitrary element of B ; recall that b now represents a special element of B . Let $c' = \sigma(c)$, choose $m \in M$ and let $m' = \pi(m)$. We need only verify that $|x + m + c| > |c|$ precisely when $|y + m' + c'| > |c'|$. Since $c' = \sigma(c)$, $|c| = |c'|$; denote the common value by λ . Notice that the case that $\lambda > |x|$ precludes $|x + m + c| > |c|$ because $|x + m| \leq |x|$. Likewise, $|y + m' + c'| > |c'|$ is precluded in this case. Thus we may assume that $\lambda \leq |x|$. If $\lambda < |x|$, $|x + m + c| > |c|$ exactly when $|m + c| > |c|$. But $|m + c| > |c|$ if and only if $|m' + c'| > |c'|$, and the latter is true exactly when $|y + m' + c'| > |c'|$ in case $\lambda < |x|$. Therefore, we may assume that $\lambda = |x|$. In this case, $|x + m + c| > \lambda$ is equivalent to $|m + c - b| > \lambda$ and $|y + m' + c'| > \lambda$ is equivalent to $|m' + c' - b'| > \lambda$. But $|m + c - b| > \lambda$ and $|m' + c' - b'| > \lambda$ fall or stand together because if either is valid then $\lambda = |c - b| = |c' - b'|$ unless $|m| > \lambda$. Moreover, the case $|m| > \lambda$ is trivial. This completes the verification that (3) remains valid for the extension of π to $\langle x \rangle \oplus M$. Now, we show that π can also be successfully extended in the alternate case.

Case 2. $|x + N| = |x|$. If $|x + m + N| > |x|$ for some $m \in M$ where $|m| = |x|$ then because $x + m$ is also orthogonal to M we can return to Case 1 by replacing x by $x + m$. Therefore, we may assume that if $m \in M$ and $|m| = |x|$ it is the case that $|x + m + N| = |x|$. Now, let $\phi(x + N) = y + N'$ where $|y| = |x|$. Suppose that $|y + m'| > |y|$ for some $m' \in M'$ and let $\pi(m) = m'$. It follows at once that $\phi(x + m + N) = y + m' + N'$ since $\phi(m + N) = m' + N'$. Thus $|x + m + N| > |x|$, but this has already been ruled out since $|m| = |m'| = |y| = |x|$. Therefore, $|y + m'| \leq |y|$ for each $m' \in M'$, and $\langle y, M' \rangle = \langle y \rangle \oplus M'$. Again, we extend π by mapping x onto y . The verification of conditions (1)–(3) for the extended map is similar to Case 1 except that $|x + m + c| > |c| = |x|$ for $c \in B$ does not occur, nor does $|y + m' + c'| > |c'| = |y|$, under the present hypothesis.

Due to the symmetry between F and F' it follows from the above local extension of π that $\pi: M \twoheadrightarrow M'$ satisfying conditions (1)–(3) can always be extended, by a sequence of local extensions of π and π^{-1} , to an isomorphism $F_K \twoheadrightarrow F'_K$ that still satisfies conditions (1)–(3), where K is a countable extension of the set J in I . Thus if J is chosen maximal then $J = I$, and the theorem is proved.

COROLLARY 1. *If N and N' are isomorphic nice subspaces of a free valuated vector space F with isomorphic quotients F/N and F/N' , then N and N' are equivalent subspaces of F in the sense that there is an automorphism of F that maps N onto N' .*

COROLLARY 2. *Let N be a nice subspace of a free valuated space F . If the structure of the valuated space $V = F/N$ can be determined by certain cardinal numbers, then the same is true of N .*

PROOF. In view of the theorem, the space N is determined up to isomorphism by a basic subspace B of N together with $V = F/N$.

The basic subspace B is determined by cardinal numbers since it is free.

REMARK. The preceding corollary is stated somewhat loosely. But when we say that a valuated vector space V is determined by certain cardinal numbers what we actually mean is that V is determined, up to isomorphism, by a function or a sequence of functions from Γ to the cardinal numbers.

For our next result we specialize to the case that Γ is well ordered although this does not yield the most general result. Let $V = \prod_{\gamma \in \Gamma} H_\gamma$ be the product of homogeneous spaces H_γ of value γ , which means that the value of each nonzero element of H_γ is γ . Among such products, V is determined by the cardinal-valued function $f(\gamma) = \dim(H_\gamma)$. This observation leads to the following result; it is of interest only in the case that N does not have a basis.

COROLLARY 3. *Suppose that Γ is well ordered and that $F/N = \prod_{\gamma \in \Gamma} H_\gamma$, where H_γ is homogeneous of value γ . As before, assume that N is a nice subspace of the free valuated space F . Up to isomorphism, N is uniquely determined by the cardinal-valued functions $f(\gamma) = \dim(H_\gamma)$ and $g(\gamma) = \dim(B_\gamma)$, where $\sum_{\gamma \in \Gamma} \oplus B_\gamma$ is a basic subspace of N with B_γ being homogeneous of value γ .*

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