EXISTENCE OF SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION

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ABSTRACT. A criterion is proved for the existence of at least one solution to the equation $u'' + u = g(u) + h$ with $u(0) = u(\pi) = 0$, where $h \in L_2[0, \pi]$ and g is continuous monotone nonincreasing.

In a number of papers [5–7] we have considered the problem of the periodic solutions (harmonics) of Liénard systems, and we have given sufficient conditions for the existence of such solutions in a number of rather general situations. This we have done by using variants of the alternative method (cf. also [1, 2, 3, 4, 8]). We are interested here in the solutions of the nonlinear problem

$$(1) \quad u'' + u = g(u) + h, \quad u(0) = u(\pi) = 0, \quad |g(u)| \leq C_1 + C_2 |u|, \\ h \in L_2[0, \pi], \quad g \text{ continuous and monotone, } g(\infty) = \infty.$$

A paper has recently appeared [10] in which the authors use a method which could be thought of as a variant of the alternative method we have used in [4–6] and show the existence of solutions of the nonlinear problem (1) under suitable growth hypotheses on $g(u)$ including oddness and a bound on C_2 .

In the present paper we should separate the case of $g(u)$ nondecreasing from the case in which $g(u)$ is nonincreasing.

For g nondecreasing neither oddness nor any bound on g is needed. In an independent paper we discuss this case among other aspects of boundary value problems at resonance. Indeed, following our results in [12] and using the ideas in [5–9], we improve this case to essentially requiring that g be eventually increasing and thereby improve on previous work in [11].

For g nonincreasing, certainly a bound on C_2 is needed since for $C_2 = 3$, $C_0 = 0$, $g(u) = -3u$, $h = \sin 2t$, problem (1) has no solution. However, for this case, g monotone nonincreasing by using the same method above, we prove Theorems (1.i) and (2.i) which seem to improve the results in [10] since oddness is not needed and the estimate $\gamma < 0.443$ in (1.i) replaces a weaker estimate in [10] (cf. Remark 2 at the end of the present paper).

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1. We first consider the following situation which seems to be relevant.

(1.i) THEOREM. Let $h: [0, \pi] \rightarrow \mathbf{R}$, $h \in L_2[0, \pi]$, $g: \mathbf{R} \rightarrow \mathbf{R}$ be continuous monotone nonincreasing with $|g(\xi)| \leq C + \gamma |\xi|$ for some constants $\gamma > 0$, $C \geq 0$, and such that $\limsup_{\xi \rightarrow \infty} (-g(\xi)/\xi) = \gamma$ in the sense that for every $\varepsilon > 0$ there are numbers $\bar{X} > 0$ as large as we want such that $-g(\bar{X})/\bar{X} > \gamma - \varepsilon$ and $-g(-\bar{X})/(-\bar{X}) > \gamma - \varepsilon$. Then, for $\gamma < 0.443$ there is at least one solution $u(t)$, $0 \leq t \leq \pi$, to problem (1), (u, u') absolutely continuous with $u'' \in L_2[0, \pi]$.

PROOF. Under the stated assumptions, we see that for every $\varepsilon > 0$ there are also numbers $\bar{\xi} > 0$ as large as we want such that $-g(\bar{\xi})/\bar{\xi} \geq \gamma - \varepsilon$, $-g(-\bar{\xi})/(-\bar{\xi}) \geq \gamma - \varepsilon$.

In terms of the alternative method (cf. e.g. [3]), we take $X = Y = L_2[0, \pi]$, $Eu = u'' + u$ for $u \in D[E]$ and boundary conditions $u(0) = u(\pi) = 0$, so that the eigenvalues of E are the integers $1 - k^2$, and the corresponding normalized eigenfunctions are $(2/\pi)^{1/2} \sin kt$, $k = 1, 2, \dots$, and $\ker E = \{\sin t\}$.

Any element $u \in L_2[0, \pi]$ has Fourier series

$$u = \sum_1^\infty a_k (2/\pi)^{1/2} \sin kt, \quad a_k = (u, (2/\pi)^{1/2} \sin kt),$$

and we define the projection operators $P: X \rightarrow X$, $Q: Y \rightarrow Y$ by taking $Pu = Qu = a_1(2/\pi)^{1/2} \sin t$. Then, for $X_1 = (I - P)X$, $Y_1 = (I - Q)Y$, and any element $u \in Y_1$, we define as usual $H: Y_1 \rightarrow X_1$ by taking

$$u_1 = Hu = - \sum_2^\infty (k^2 - 1)^{-1} a_k (2/\pi)^{1/2} \sin kt$$

for

$$u = \sum_2^\infty a_k (2/\pi)^{1/2} \sin kt \in Y_1.$$

Then

$$\begin{aligned} \|u\| &= \left(\sum_2^\infty a_k^2 \right)^{1/2}, \\ \|Hu\| &= \left(\sum_2^\infty (k^2 - 1)^{-2} a_k^2 \right)^{1/2} \leq 3^{-1} \left(\sum_2^\infty a_k^2 \right)^{1/2} = 3^{-1} \|u\|, \\ \|Hu\|_\infty &\leq \sup_t \left(\sum_2^\infty (k^2 - 1)^{-2} \right)^{1/2} (2/\pi)^{1/2} \left(\sum_2^\infty a_k^2 \sin^2 t \right)^{1/2} \\ &\leq \mu \left(\sum_2^\infty a_k^2 \right)^{1/2} = \mu \|u\|, \end{aligned}$$

where $\mu = (2/\pi)^{1/2}\sigma$ and $\sigma = (\sum_2^\infty (k^2 - 1)^{-2})^{1/2}$. It is easy to compute σ by comparison with the well-known series $S_4 = \sum_1^\infty k^{-4} = 1.08232$. Indeed, for every m we have

$$\sum_2^{m-1} (k^2 - 1)^{-2} + \left[S_4 - \sum_1^{m-1} k^{-4} \right] < \sigma^2 \\ < \sum_2^{m-1} (k^2 - 1)^{-2} + (m^4(m^2 - 1)^{-2}) \left(S_4 - \sum_1^{m-1} k^{-4} \right),$$

and for $m = 10$ we obtain $\sigma = 0.36736$ and $\mu = 0.29311$ (the error being $< 10^{-5}$). With these notations, problem (1), or $Ex = Nx$, is equivalent to the system of auxiliary and determining equations

$$(2) \quad u_1 = K_1 u \equiv H(I - P)Nu, \quad K_0 u \equiv QNu = 0,$$

where $u = u^* + u_1$, $u^* \in X_0$, $u_1 \in X_1$. We will now apply an abstract existence theorem derived in [13] to the above system of equations.

Here N is continuous, $H(I - P)$ is compact; hence, $K_1 u = H(I - P)N$ is compact. On the other hand, $K_0 = QN$ is continuous. Let us now prove that we can take R_0 , $r > 0$, so that $\|u^*\| \leq R_0$, $\|u_1\| \leq r$, $u = u^* + u_1$ implies $\|K_1 u\| \leq r$. For any R_0 , $r > 0$, let Ω denote the set $\Omega = S_0 \times S_1$ with $S_0 = [u^* \in X_0 \mid \|u^*\| \leq R_0]$, $S_1 = [u_1 \in X_1 \mid \|u_1\| \leq r]$.

First, we have

$$|(Nu)(t)| = |g(u) + h| \leq C + \gamma |u(t)| + |h(t)|, \\ \|Nu\| \leq C\sqrt{\pi} + \gamma \|u\| + \|h\|, \\ \|\bar{u}_1\| = \|H(I - P)Nu\| \leq 3^{-1}C\sqrt{\pi} + 3^{-1}\gamma(R_0 + r) + 3^{-1}\|h\|,$$

and thus $\|K_1 u\| \leq r$ provided

$$\gamma < 3, \quad \text{and} \quad r \geq (1 - 3^{-1}\gamma)^{-1} (3^{-1}C\sqrt{\pi} + 3^{-1}\gamma R_0 + 3^{-1}\|h\|), \quad \text{or} \\ (3) \quad r \geq pR_0 + q, \quad \text{with} \\ p = (1 - 3^{-1}\gamma)^{-1} (3^{-1}\gamma), \quad q = (1 - 3^{-1}\gamma)^{-1} (3^{-1}C\sqrt{\pi} + 3^{-1}\|h\|).$$

We have just proved that $\|\bar{u}_1\| \leq r$ for $\|u^*\| \leq R_0$, $\|u_1\| \leq r$. We shall now verify that

$$(4) \quad (SQN(u^* + \bar{u}_1), u^*) \leq 0 \quad [\geq 0]$$

for all $u^* \in X_0$, $u_1 \in X_1$, $\|u^*\| = R_0$, $\|u_1\| \leq r$. When this is done, we know by (2.i) of [13] that there is at least one point $u = u^* + u_1 \in \Omega$ such that $u_1 = K_1 u$, $K_0 u = 0$, that is, $u = u^* + u_1$ is a solution of the original problem (1). Statement (2.i) is proved in [13] by the Leray-Schauder topological argument, and an alternative proof is also given in [13] in terms of Schauder's fixed point theorem.

Let $g_1 = -g$. Condition (4) in the present case becomes, for $\rho = \pm R_0$ and dropping the factor ρ ,

$$(5) \quad I = \int_0^\pi [g_1(\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t)) - h(t)] (2/\pi)^{1/2} \sin t \, dt \\ \geq 0 \quad \text{for } \rho = R_0,$$

and $I \leq 0$ for $\rho = -R_0$. Let us see in detail the first case. For some η , $0 < \eta < (2/\pi)^{1/2}$, let $F_0 = [t \in [0, \pi] \mid (2/\pi)^{1/2} \sin t > \eta]$, and $F_1 = [0, \pi] - F_0$. We have now

$$|g_1(\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t))| \leq C + \gamma |\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t)|, \quad t \in [0, \pi],$$

where

$$\|u\| = \|\rho(2/\pi)^{1/2} \sin t + \overline{u_1}\| \leq (R_0^2 + r^2)^{1/2},$$

and hence

$$\|g_1(\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t))\| \leq C\sqrt{\pi} + \gamma(R_0^2 + r^2)^{1/2}, \\ \|\overline{u_1}\|_\infty = \|H(I - P)[g_1(\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t)) - h]\|_\infty \\ \leq \mu(C\sqrt{\pi} + \gamma R_0 + \gamma r + \|h\|).$$

Therefore, for $t \in F_0$ we also have

$$(6) \quad \rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t) \geq \eta R_0 - \mu\sqrt{\pi} C - \mu\gamma R_0 - \mu\gamma r - \mu\|h\| = \chi, \quad t \in F_0.$$

To make $\chi > 0$, we require

$$(7) \quad \eta - \mu\gamma > 0, \quad \text{and} \quad (\eta - \mu\gamma)R_0 > \mu\sqrt{\pi} C + \mu\gamma r + \mu\|h\|, \quad \text{or} \\ \eta - \mu\gamma > 0, \quad R_0 > p_1 r + q_1, \quad \text{with} \\ p_1 = (\eta - \mu\gamma)^{-1}(\mu\gamma), \quad q_1 = (\eta - \mu\gamma)^{-1}(\mu\sqrt{\pi} C + \mu\|h\|).$$

We note that, for $0 < \sigma < 1$, $\sin t = \sigma$, $0 < t < \pi/2$, we take $t = \arcsin \sigma = \sigma\kappa(\sigma)$, where $1 < \kappa(\sigma) < \pi/2 = 1.57079$ and $\kappa(\sigma) \rightarrow 1$ as $\sigma \rightarrow 0+$. Thus, for $0 < \eta < (2/\pi)^{1/2}$ we have

$$(2/\pi)^{1/2} \sin t = \eta \quad \text{for } \sin t = (\pi/2)^{1/2} \eta,$$

or

$$t = (\pi/2)^{1/2} \eta \kappa \quad \text{with } \kappa = \kappa((\pi/2)^{1/2} \eta) = ((\pi/2)^{1/2} \eta)^{-1} \arcsin((\pi/2)^{1/2} \eta).$$

Also,

$$\text{meas } F_0 = \pi - 2(\pi/2)^{1/2} \eta \kappa, \quad \text{meas } F_1 = 2(\pi/2)^{1/2} \eta \kappa,$$

with $\kappa = \kappa((\pi/2)^{1/2}\eta)$. Finally, since g_1 is monotone increasing, we have

$$(8) \quad \int_{F_0} \left[g_1 \left(\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t) \right) \right] (2/\pi)^{1/2} \sin t \, dt \\ \geq (\pi - 2(\pi/2)^{1/2}\eta\kappa) \eta g_1(\chi).$$

On the other hand, for $t \in F_1$, we have

$$\begin{aligned} |\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t)| &\leq R_0\eta + \mu(C\sqrt{\pi} + \gamma R_0 + \gamma r + \|h\|), \quad t \in F_1, \\ |g_1[\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t)]| &\leq C + \gamma R_0\eta + \mu\gamma C\sqrt{\pi} + \mu\gamma^2 R_0 + \mu\gamma^2 r + \mu\gamma\|h\|, \quad t \in F_1, \\ (9) \quad \left| \int_{F_1} \left[g_1 \left(\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t) \right) \right] (2/\pi)^{1/2} \sin t \, dt \right| \\ &\leq 2(\pi/2)^{1/2} \kappa \eta^2 \cdot (C + \gamma R_0\eta + \mu\gamma C\sqrt{\pi} + \mu\gamma^2 R_0 + \mu\gamma^2 r + \mu\gamma\|h\|). \end{aligned}$$

From the assumptions on g_1 we know that given $\varepsilon > 0$, there are numbers $\bar{\chi} > 0$ as large as we want such that $g_1(\bar{\chi}) \geq (\gamma - \varepsilon)\bar{\chi}$. Thus, if χ , as defined by (6), is one of these numbers, then by (6) and the monotonicity of g_1 we have, for $t \in F_0$.

$$\begin{aligned} g_1(\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t)) &\geq g_1(\bar{\chi}) \\ &\geq (\gamma - \varepsilon)(\eta R_0 - \mu\sqrt{\pi}C - \mu\gamma R_0 - \mu\gamma r - \mu\|h\|). \end{aligned}$$

Finally, from (5), (8), (9) we have

$$\begin{aligned} (10) \quad I &= \left(\int_{F_0} + \int_{F_1} \right) g_1(\rho(2/\pi)^{1/2} \sin t + \overline{u_1}(t)) (2/\pi)^{1/2} \sin t \, dt \\ &\quad - \int_0^\pi h(t) (2/\pi)^{1/2} \sin t \, dt \\ &\geq (\pi - 2(\pi/2)^{1/2}\eta\kappa) \eta (\gamma - \varepsilon) (\eta R_0 - \mu\sqrt{\pi}C - \mu\gamma R_0 - \mu\gamma r - \mu\|h\|) \\ &\quad - 2(\pi/2)^{1/2} \kappa \eta^2 (C + \gamma R_0\eta + \mu\gamma C\sqrt{\pi} + \mu\gamma^2 R_0 + \mu\gamma^2 r + \mu\gamma\|h\|) - \|h\| \\ &= \left[(\pi - 2(\pi/2)^{1/2}\eta\kappa) \eta (\gamma - \varepsilon) (\eta - \mu\gamma) - 2(\pi/2)^{1/2} \kappa \eta^2 (\gamma\eta + \mu\gamma^2) \right] R_0 \\ &\quad - (\pi - 2(\pi/2)^{1/2}\eta\kappa) \eta (\gamma - \varepsilon) (\mu\sqrt{\pi}C + \mu\gamma r + \mu\|h\|) \\ &\quad - 2(\pi/2)^{1/2} \kappa \eta^2 (C + \mu\gamma\sqrt{\pi}C + \mu\gamma^2 r + \mu\gamma\|h\|) - \|h\|. \end{aligned}$$

To make $I > 0$, first we require the coefficient m of R_0 to be positive, hence

$$(11) \quad m = (\pi - 2(\pi/2)^{1/2}\eta\kappa) \eta (\gamma - \varepsilon) (\eta - \mu\gamma) - 2(\pi/2)^{1/2} \kappa \eta^2 (\gamma\eta + \mu\gamma^2) > 0,$$

and by taking $\alpha = 2(\pi/2)^{1/2}\kappa$, $\varepsilon = 0$, and dividing by $\eta\gamma$, we require

$$(12) \quad (\pi - \alpha\eta)(\eta - \mu\gamma) > \alpha\eta(\eta + \mu\gamma), \quad \text{or} \quad \eta - \mu\gamma > (2\alpha/\pi)\eta^2.$$

In other words, not only do we need $\eta - \mu\gamma > 0$ as required in (7), but we also need $\eta - \mu\gamma > (2\alpha/\pi)\eta^2$ or, equivalently, $\eta - (2\alpha/\pi)\eta^2 > \mu\gamma$. The first member takes its maximum value for $\eta = \eta_0 = \pi/4\alpha$, and thus γ must be below the corresponding value, or $\gamma < \gamma_0 = \mu^{-1}\eta_0(1 - (2\alpha/\pi)\eta_0) = \pi/8\alpha\mu = (2\mu)^{-1}\eta_0$. By taking note of the value of α we also have

$$\eta_0 = \pi/4\alpha = \pi(8(\pi/2)^{1/2}\kappa)^{-1} = (\pi\eta_0/8)(\arcsin((\pi/2)^{1/2}\eta_0))^{-1},$$

and we see that this relation determines η_0 , since $0 < \eta_0 < (2/\pi)^{1/2}$ and $\arcsin((\pi/2)^{1/2}\eta_0) = \pi/8$. For $\arcsin c = \pi/8 = 0.39269$, we then have $c = \sin(0.39269) = 0.38267$, so that $\kappa = (0.39269)(0.38267)^{-1} = 1.02618$, and then

$$\eta_0 = (\sqrt{2}/\sqrt{\pi})(0.38267) = 0.30533, \quad \alpha = (2(\pi/2)^{1/2}\kappa) = 2.57224,$$

and

$$(13) \quad \gamma_0 = (0.30533) \cdot 2^{-1}(0.29311)^{-1} = 0.52084, \quad 0 < \gamma < \gamma_0.$$

We see that we can take $\eta = \eta_0$ and then $0 < \gamma < \gamma_0$, so that (12) holds. Hence, we can take $\varepsilon > 0$ sufficiently small so that (11) holds, that is, the coefficient m of R_0 in the estimate (10) of I is positive. For $\varepsilon = \varepsilon'\gamma$, we also have from (11):

$$\begin{aligned} m &= (\pi - \alpha\eta)\eta\gamma(\eta - \mu\gamma) - \alpha\eta^2(\gamma\eta + \mu\gamma^2) - \varepsilon'(\pi - \alpha\eta)\eta\gamma(\eta - \mu\gamma) \\ &= \eta\gamma[\pi(\eta - \mu\gamma) - 2\alpha\eta^2 - \varepsilon'(\pi - \alpha\eta)(\eta - \mu\gamma)] > 0. \end{aligned}$$

Then, for the positivity of I , we require from (10) that

$$\begin{aligned} &\eta\gamma[\pi(\eta - \mu\gamma) - 2\alpha\eta^2 - \varepsilon'(\pi - \alpha\eta)(\eta - \mu\gamma)]R_0 \\ &\geq \eta\gamma[(\pi - \alpha\eta)(1 - \varepsilon')\mu\gamma + \alpha\eta\mu\gamma]r \\ &+ \eta\gamma[(\pi - \alpha\eta)(1 - \varepsilon')(\mu\sqrt{\pi}C + \mu\|h\|) + \alpha\eta(\gamma^{-1}C + \mu\sqrt{\pi}C + \mu\|h\|)] + \|\eta\gamma\|, \end{aligned}$$

or

$$(14) \quad R_0 \geq p_2r + q_2,$$

with

$$\begin{aligned} p_2 &= [\pi(\eta - \mu\gamma) - 2\alpha\eta^2 - \varepsilon'(\pi - \alpha\eta)(\eta - \mu\gamma)]^{-1}(\mu\gamma)[(\pi - \alpha\eta)(1 - \varepsilon') + \alpha\eta], \\ q_2 &= [\pi(\eta - \mu\gamma) - 2\alpha\eta^2 - \varepsilon'(\pi - \alpha\eta)(\eta - \mu\gamma)]^{-1} \\ &\times [(\pi - \alpha\eta)(1 - \varepsilon')(\mu\sqrt{\pi}C + \mu\|h\|) \\ &+ \alpha\eta(\gamma^{-1}C + \mu\sqrt{\pi}C + \mu\|h\|) + (\eta\gamma)^{-1}\|\eta\gamma\|]. \end{aligned}$$

Relations (3), (7) and (14) require that

$$\begin{aligned} (15) \quad R_0 &> p_1r + q_1 > p_1pR_0 + (p_1q + q_1), \\ R_0 &> p_2r + q_2 > p_2pR_0 + (p_2q + q_2). \end{aligned}$$

Hence, we require $p_1 p < 1$, $p_2 p < 1$, and

$$(16) \quad R_0 > (1 - p_1 p)^{-1}(p_1 q + q_1), \quad R_0 > (1 - p_2 p)^{-1}(p_2 q + q_2),$$

and we require r to satisfy (15), which now implies (3).

We now have

$$\begin{aligned} p_1 p &= (\eta - \mu\gamma)^{-1}(\mu\gamma)(3^{-1}\gamma)(1 - 3^{-1}\gamma)^{-1}, \\ p_2 p &= [\pi(\eta - \mu\gamma) - 2\alpha\eta^2 - \varepsilon'(\pi - \alpha\eta)(\eta - \mu\gamma)]^{-1} \\ &\quad \times \mu\gamma[(\pi - \alpha\eta)(1 - \varepsilon') + \alpha\eta](3^{-1}\gamma)(1 - 3^{-1}\gamma)^{-1}. \end{aligned}$$

To guarantee that $p_1 p < 1$, $p_2 p < 1$, we first take $\varepsilon' = 0$ so that these requirements reduce to

$$\mu\gamma(3^{-1}\gamma) < (\eta - \mu\gamma)(1 - 3^{-1}\gamma), \quad \text{or} \quad (3^{-1}\eta + \mu)\gamma - \eta < 0,$$

and

$$\mu\gamma(\pi)(3^{-1}\gamma) < (\pi(\eta - \mu\gamma) - 2\alpha\eta^2)(1 - 3^{-1}\gamma)$$

or

$$(3^{-1}\eta + \mu)\gamma - \eta < 0, \quad [3^{-1}\eta(\pi - 2\alpha\eta) + \pi\mu]\gamma - \eta(\pi - 2\alpha\eta) < 0.$$

We have $\mu = 0.29311$, and we take $\eta = \eta_0 = 0.30533$, so that

$$\begin{aligned} \kappa &= 1.02612, \quad \alpha = 2.57224, \quad 3^{-1}\eta + \mu = 0.39488, \quad \pi - 2\alpha\eta = 1.57083, \\ 3^{-1}\eta(\pi - 2\alpha\eta) + \pi\mu &= 1.08070, \quad \eta(\pi - 2\alpha\eta) = 0.47962, \\ (3^{-1}\mu)(2\alpha\eta) &= 0.15347, \end{aligned}$$

and the inequalities above become, respectively,

$$(0.39488)\gamma - 0.30533 < 0, \quad (1.08070)\gamma - 0.47962 < 0.$$

Hence, $\gamma < \gamma_{11} = 0.77322$, and $\gamma < \gamma_{21} = 0.4438$, the latter restriction being stronger than the former, and also stronger than the two others, $\gamma < 3$ and $\gamma < \gamma_0 = 0.52084$, we met before. We take $\gamma = 0.443$. By choosing ε , that is, $\varepsilon' > 0$ sufficiently small, we have $p_1 p < 1$, $p_2 p < 1$, and (3), (7) and (14) reduce to (15) and (16).

It is enough that we now take $R_0 > 0$ sufficiently large and in such a way that the corresponding χ as given by (6) is one of the numbers $\bar{\chi}$ which are now determined since we have already chosen $\varepsilon = \gamma\varepsilon' > 0$. For $\rho = -R_0$ the same considerations hold with (6) replaced by

$$\rho(2/\pi)^{1/2}\sin t + \overline{u_1}(t) \leq -\eta R_0 + \mu\sqrt{\pi}C + \mu\gamma R_0 + \mu\gamma r + \mu\|h\| = -\chi, \quad t \in F_0.$$

By the statement (2.i) of [13] we conclude that there is some $u = u^* + u_1 \in \Omega$ such that $u_1 = K_1 u$, $K_0 u = 0$. This completes the proof of (1.i).

2. We shall consider here another situation which also seems to be of interest.

(2.i) THEOREM. Let $h: [0, \pi] \rightarrow \mathbf{R}$, $h \in L_2[0, \pi]$ and let $g: \mathbf{R} \rightarrow \mathbf{R}$ be continuous monotone with $|g(\xi)| \leq C + D|\xi|$ for all $\xi \in \mathbf{R}$ and some constants $C \geq 0$ and $0 < \gamma \leq D$, and such that $\limsup_{\xi \rightarrow \infty} (-g(\xi)/\xi) = \gamma$ in the same sense as in Theorem (1.i). Then, for any $0 < \gamma < 0.443$, there is a constant $\Gamma = \Gamma(\gamma) > 0.443$, such that, for $\gamma \leq D < \Gamma(\gamma)$, problem (1) has at least a solution $u(t)$, $0 \leq t \leq \pi$.

For $D = \gamma$ this theorem reduces to Theorem (1.i).

PROOF. Let $D = \theta\gamma$ for some $\theta \geq 1$. Let us proceed as in §1, where now in the estimates we shall need $D = \theta\gamma$ instead of γ , while the estimates where we have $\gamma - \varepsilon$ remain the same. The final result is that the problem has solutions if we can satisfy certain inequalities, where now θ appears together with γ . For $\theta = 1$ these inequalities reduce to those of §1, and we have seen that the latter could be satisfied in the strong sense by taking η, κ, α as in §1 and $\gamma < 0.443$. By continuity argument there is some $\theta_0 > 1$, depending on γ , such that the same inequalities are satisfied for all θ , $1 \leq \theta \leq \theta_0$.

REMARK 1. (1.i) and (2.i) do not hold for $\gamma = 0$. Indeed, for $g \equiv 0$, $h = \varepsilon \sin t$, $\varepsilon > 0$, problem (1) has no solution (no matter how small ε is).

However, if g is continuous in $\mathbf{R}$ with $\|g(\xi)\| \leq M$ for $\xi \in \mathbf{R}$ (and thus $\gamma = 0$), and if there are constants $c < d$, $C < D$ such that $g(\xi) \leq C$ for $\xi \leq c$, $g(\xi) \geq D$ for $\xi \geq d$, and $|\int_0^\pi h(t) \sin t \, dt| < D - C$, then there is a solution $x(t)$, $0 \leq t \leq \pi$, to problem (1).

This is the essence of the Lazer and Leach theorem [9], which is restated here in the present situation. The proof, on the same idea as the present paper, is given in [3, p. 144] and can be repeated in the present case.

REMARK 2. Statement (1.i) seems to improve those in [10] where it is required that g be odd (we do not need this assumption) and $\gamma < 0.24347$ (we require $\gamma < 0.443$). For instance, for $g(\xi) = 1 + 3^{-1}\xi$, for $\xi > 0$, $g(\xi) = e^\xi + 3^{-1}\xi$ for $\xi \leq 0$, and $h(t) = \sin t$, problem (1) has a nonzero solution.

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