ON A SINGULAR ELLIPTIC EQUATION

WEI-MING NI¹

ABSTRACT. In this paper, we study the singular elliptic equation $Lu + K(x)u^p = 0$, where L is a uniformly elliptic operator of divergence form, $p > 1$ and $K(x)$ has a singularity at the origin. We prove that this equation does not possess any positive (local) solution in any punctured neighborhood of the origin if there exist two constants C_1, C_2 such that $C_1|x|^\sigma \geq K(x) \geq C_2|x|^\sigma$ near the origin for some $\sigma \leq -2$ (with no other condition on the gradient of K). In fact, an integral condition is derived.

1. Introduction. In recent years, there has been some interest in studying the following singular nonlinear elliptic equation

$$(1) \quad \Delta u + h(x)u^p = 0$$

near the origin in $\mathbf{R}^n$, $n \geq 3$, where $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$, $p > 1$, and h is nonnegative which may have the origin as an isolated singularity (see, for example, [A], [G, S], [L] and the references therein). In [A], [G, S], h is assumed to satisfy the following conditions (in addition to being positive) near the origin.

$$(2) \quad C_1|x|^\sigma \geq h(x) \geq C_2|x|^\sigma \quad \text{for some constants } C_1, C_2 > 0 \text{ and } \sigma \in \mathbf{R},$$

$$(3) \quad |\nabla \log h(x)| \leq C_3/|x| \quad \text{for some constant } C_3 > 0.$$

Indeed, the asymptotic behavior of positive singular solutions of (1) near the origin are obtained in case $\sigma > -2$ and p in some appropriate ranges (cf. [A], [G, S] and [L]). It is also shown in [G, S] that if $\sigma < -2$, h satisfies (2) and (3) and $1 < p < (n+2)/(n-2)$, then equation (1) does not possess any positive solution in $\Omega \setminus \{0\}$, where Ω is any neighborhood of the origin. The purpose of this present note is to show that *equation (1) does not possess any positive solution in $\Omega \setminus \{0\}$ for any $p > 1$ provided h satisfies (2) and $\sigma \leq -2$* , i.e. we obtain the same conclusion for all $p > 1$ without any condition on ∇h . Moreover, the exponent $\sigma \leq -2$ is best possible (cf. [A]). In fact, this follows from our main result which applies to more general elliptic operators than Laplacian and the condition on h which insures the above conclusion is an integral condition which actually allows h being "oscillatory" near the origin. We also mention that our proof is simple and elementary in contrast to the complicated proof in [G, S].

Received by the editors October 18, 1982.

1980 *Mathematics Subject Classification.* Primary 35J60.

Key words and phrases. Singular elliptic equation, nonexistence.

¹Partially supported by NSF Grant MCS-8200033 and a Summer Research Appointment from the Graduate School of the University of Minnesota.

©1983 American Mathematical Society
0002-9939/83/0000-1315/\$01.75

2. The main result. The equation we are studying is

$$(4) \quad Lu + K(x)u^p = 0$$

in $\Omega \setminus \{0\}$, where Ω is an arbitrary neighborhood of the origin in $\mathbf{R}^n$, $n \geq 3$; $p > 1$, $K > 0$ is smooth in $\Omega \setminus \{0\}$ and

$$L \equiv \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

where $a_{ij} \in C^2(\Omega)$, is uniformly elliptic in Ω (we should point out that the conditions on (a_{ij}) may be relaxed without causing any further work later on; see, for example, §2 in [A]).

Let $G(x)$ be the Green's function of L on Ω with singularity at the origin; we then define

$$\tilde{K}(t) = \min_{x \in \Gamma(t)} K(x)$$

where $\Gamma(t) = \{x \in \Omega \mid G(x) = t\}$. Now, we can state our main result.

THEOREM. *Under the above hypotheses on L and K equation (4) does not possess any positive solution in any punctured neighborhood of the origin provided*

$$(5) \quad \int_0^\infty t^{-n/(n-2)} \tilde{K}(t) dt = \infty.$$

PROOF. The proof of our theorem uses some averaging method and a result in nonlinear oscillation theory for ordinary differential inequalities in recent papers of Ni [N] and of Aviles [A]. We define

$$\bar{u}(t) = \int_{\Gamma(t)} \frac{\sum_{i,j=1}^n a_{ij} G_{x_i} G_{x_j}}{|\nabla G(x)|} u(x) dS$$

as in [A]. Then differentiating $\bar{u}$ twice with respect to t , applying Green's theorem several times, we arrive at the following inequality,

$$(6) \quad \bar{u}_{tt} + C \frac{\tilde{K}(t)}{t^{2(n-1)/(n-2)}} \bar{u}^p(t) \leq 0$$

for $t \geq T$, where C is some (fixed) positive constant (see Lemma 1 in §2 of [A] for details). Now, applying Theorem 3.43 in [N] to (6), we conclude our proof. Q.E.D.

REMARKS. (7) Since $G(x)$ behaves like $C/|x|^{n-2}$ near the origin (by the assumption on L), we see that (5) is fulfilled if $K(x)$ behaves like $|x|^\sigma$, $\sigma \leq -2$, near $x = 0$. In fact, since only the total "weight" of K matters, K does not have to have ∞ as its limit at $x = 0$, it could have zero as its limit inf at the origin.

NOTE ADDED IN PROOF. The conclusion of Theorem 3.43 in [N] is not quite correct as it stands, what is really proved there is that *a solution cannot be ultimately positive*, which is just sufficient for the application in [N] and is just what we need in the proof of the main result here. In fact this result in nonlinear oscillation was proved earlier by H. Teufel, Jr., *A note on second-order differential equations and functional*

differential equations, appeared in Pacific J. Math. **41** (1972), 537–541, and generalized by F. V. Atkinson, see, e.g. *On second-order differential inequalities*, in Proc. Roy. Soc. Edinburgh (A) **72** (1973), 109–127. The author wishes to thank Professor Atkinson for pointing out these remarks.

REFERENCES

[A] P. Aviles, *On isolated singularities in some nonlinear partial differential equations*, Indiana Univ. Math. J. (to appear).

[G, S] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **36** (1981), 525–598.

[L] P.-L. Lions, *Isolated singularities in semilinear problems*, J. Differential Equations **38** (1980), 441–450.

[N] W.-M. Ni, *On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalizations and applications in geometry*, Indiana Univ. Math. J. **31** (1982), 493–529.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455