ON ISOCLINAL SEQUENCES OF SPHERES

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ABSTRACT. In n -dimensional inversive geometry it is possible to construct an infinite sequence of $(n - 1)$ -spheres with the property that every $n + 2$ consecutive numbers are isoclinal. For every such sequence there is a Möbius transformation which advances the spheres of the sequence by one.

1. Introduction. In the beautiful paper [1], Coxeter showed that in n -dimensional inversive space ($n \geq 2$) there is an infinite sequence of $(n - 1)$ -spheres such that every $n + 2$ consecutive members are mutually tangent. Given such sequence there is a Möbius transformation mapping the sequence onto itself (see [4]). Mauldon in [3] considered sets of spheres with the same mutual inclination, and Gerber in [2] extended these sets to infinite sequences. Here we give a recursion formula for computing coordinates of the spheres in such sequences. Using this formula we show that such a sequence can, as well as a tangent one, be mapped onto itself by a Möbius transformation.

2. Notation and definitions. In considering n -dimensional inversive geometry we can use vectors with $n + 2$ coordinates to denote points and spheres. Let Σ be a unit sphere in $\mathbf{R}^{n+1}$ centred at the origin, and Π an n -flat through the origin. Stereographic projection establishes one-to-one correspondence between Π and $\Sigma \setminus \{\text{north pole}\}$. To every point X in Π we can assign an $(n + 2)$ -tuple $(x, 1)$ where x is the coordinate of the stereographic projection of X . Similarly, to every sphere Y in Π we can assign an $(n + 2)$ -tuple $(\csc \theta \cdot y, \cot \theta)$, where y is the coordinate of the centre and θ the angular radius of the stereographic projection of Y . The coordinates are positive homogeneous.

In connection with these coordinates we introduce the Lorentz bilinear form

$$A * B = (a_1, \dots, a_{n+2}) * (b_1, \dots, b_{n+2}) = a_1 b_1 + \dots + a_{n+1} b_{n+1} - a_{n+2} b_{n+2}.$$

A vector A represents a point if and only if $A * A = 0$ and $a_{n+2} > 0$, and it names a sphere when $A * A = 1$. A point A lies on a sphere B if $A * B = 0$. If A and B are spheres then $A * B$ is called the inclination between A and B . Furthermore:

$$\begin{aligned} A * B &= \pm 1 && \text{if } A \text{ and } B \text{ are internally and externally tan-} \\ &&& \text{gent respectively.} \\ A * B &= \cos \theta && \text{if } A \text{ and } B \text{ intersect at angle } \theta. \end{aligned}$$

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$A * B = \pm \cosh \delta$ if A and B are disjoint with nested and disjoint interiors respectively, and their inversive distance is δ .

If D is a sphere such that $A - 2(D * A)D = B$ then D is a midsphere, i.e. inversion in D interchanges A and B . For proofs see [5].

3. Sets of isoclinal spheres. A set of spheres is said to be isoclinal if any two spheres from the set have the same mutual inclination. If the inclination is γ we say that the set is γ -clinal. In particular, a set of $n + 2$ tangent spheres is called a cluster. Spheres of any cluster form a basis for $(n + 2)$ -space (see [5]). Moreover,

THEOREM 1. *Any ordered set of $n + 2$ γ -clinal $(n - 1)$ -spheres, $\gamma \neq 1$, forms a basis for $(n + 2)$ -space.*

PROOF. Let $\mathfrak{B} = \{B_0, B_1, \dots, B_{n+1}\}$ be a γ -clinal set. Define a matrix $B = (b_{ij}) = (B_i * B_j)$. Then $B = (1 - \gamma)I + \gamma J$, where I is the $(n + 2) \times (n + 2)$ identity matrix and J is the $(n + 2) \times (n + 2)$ matrix with all entries 1. It is easy to verify that $(1 - \gamma)^{-1}I - (1 - \gamma)^{-1}(n + 1 + \gamma^{-1})^{-1}J = B^{-1}$, and hence B is nonsingular, so that the vectors in $\mathfrak{B}$ are linearly independent.

Clearly for any $\gamma < 1$ there is a set of $n + 1$ γ -clinal $(n - 1)$ -spheres (spheres with centres at the vertices of a simplex). If there is a set of $n + 2$ γ -clinal spheres it can be inverted into $(n + 1)$ -spheres centred at the vertices of a simplex and a sphere centred at the centre of the simplex. To see that, let $\{B_0, B_1, \dots, B_{n+1}\}$ be γ -clinal. Then $D_{0,1} = [1/2(1 - \gamma)]^{1/2}(B_0 - B_1)$ is a midsphere for B_0 and B_1 and it is perpendicular to $B_2, \dots, B_{n+2}$. Similarly we define $D_{1,2}, \dots, D_{n,0}$. It is easy to see that there is a point $P = \alpha(B_0 + \dots + B_n) + \beta B_{n+1}$ (for some α and β) such that P is on all $D_{i,i+1}$. Inversion in this point transforms $B_0, B_1, \dots, B_n$ into the spheres centred at the vertices of a simplex, and since B_{n+2} is perpendicular to all $D_{i,i+1}$ its centre must be the centre of the simplex.

The described set of $n + 2$ γ -clinal $(n - 1)$ -spheres in $\mathbf{R}^n$ may be regarded as the "equatorial" section of the set $\{\bar{B}_0, \bar{B}_1, \dots, \bar{B}_{n+1}\}$ of $n + 2$ γ -clinal n -spheres in $\mathbf{R}^{n+1}$. It is easy to see that there are exactly two nonintersecting spheres tangent to all $\bar{B}_i$, $i = 0, \dots, n + 1$. Inverting in one of the points common to all the spheres perpendicular to the two disjoint spheres, we obtain two concentric spheres. $\bar{B}_0, \dots, \bar{B}_{n+1}$ are inverted into congruent spheres sandwiched between them. Hence centres of $\bar{B}_0, \dots, \bar{B}_{n+1}$ are at the vertices of an $(n + 1)$ -simplex.

We conclude now that given two ordered sets of $n + 2$ γ -clinal spheres there is a Möbius transformation mapping one set onto the other, i.e. any two such sets are inversely equivalent.

Here we give another proof of the result due to Mauldon [3] providing necessary and sufficient conditions for the existence of $n + 2$ γ -clinal spheres.

THEOREM 2. *In n -dimensional inversive space there exists a set of $n + 2$ γ -clinal $(n - 1)$ -spheres if and only if $\gamma \leq -(n + 1)^{-1}$.*

PROOF. If there is a set of $n + 2$ γ -clinal $(n - 1)$ -spheres $\{B_0, B_1, \dots, B_{n+1}\}$ we can think of it as a set of congruent n -caps on an n -sphere whose centres are vertices $c_0, c_1, \dots, c_{n+1}$, respectively, of an $(n + 1)$ -simplex inscribed in Σ . Let $C_0, C_1, \dots, C_{n+1}$ be congruent n -caps forming a cluster with their centres at $c_0, c_1, \dots, c_{n+1}$, respectively. Then

$$B_j = aC_i + b \sum_{\substack{j=0 \\ j \neq i}}^{n+1} C_j, \quad i \in \{0, 1, \dots, n + 1\}.$$

Since

$$B_i * B_j = \begin{cases} \gamma & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

we have

$$\begin{aligned} a^2 + 2ab(n - 1) + b^2(n^2 + 1) &= \gamma, \\ a^2 - 2ab(n + 1) - b^2(n^2 - 1) &= 1. \end{aligned}$$

It is easy to see that a and b are real if and only if $\gamma \leq -(n + 1)^{-1}$.

THEOREM 3. Given a set $\{B_0, B_1, \dots, B_{n+1}\}$ of γ -clinal $(n - 1)$ -spheres where $\gamma \neq -n^{-1}$, $-(n + 1)^{-1}$, there is exactly one sphere $B_{n+2} \neq B_0$ such that $\{B_1, B_2, \dots, B_{n+2}\}$ is γ -clinal and

$$(1) \quad B_{n+2} = -B_0 + \frac{2}{n + \gamma^{-1}} (B_1 + \dots + B_{n+1}).$$

When $\gamma = -n^{-1}$, $-(n + 1)^{-1}$ there is no sphere with this property.

PROOF. Any sphere B_{n+2} such that $B_{n+2} * B_i = \gamma$ for all $i \in \{1, \dots, n + 1\}$ must be of the form $aB_0 + b(B_1 + \dots + B_{n+1})$ where

$$a + b(n + \gamma^{-1}) = 1, \quad a^2 + 2ab\gamma(n + 1) + b^2\gamma(n + 1)(n + \gamma^{-1}) = 1.$$

These equations have exactly two solutions, $a = 1$, $b = 0$ yielding B_0 , and $a = -1$, $b = 2(n + \gamma^{-1})^{-1}$ if $\gamma \neq -n^{-1}$, $-(n + 1)^{-1}$.

THEOREM 4. Given a set $\{B_0, B_1, \dots, B_{n+1}\}$ of γ -clinal $(n - 1)$ -spheres there is exactly one sphere D_0 perpendicular to $B_1, \dots, B_{n+1}$ if and only if $\gamma < -n^{-1}$. There is none otherwise.

PROOF. It is easy to see that $D_0 = -a(n + \gamma^{-1})B_0 + a(B_1 + \dots + B_{n+1})$ with $a^2 = (1 - \gamma)^{-1}(n + \gamma^{-1})^{-1}(n + 1 + \gamma^{-1})^{-1}$. Here $a^2 > 0$ if and only if $\gamma < -n^{-1}$.

COROLLARY 1. If D_0 is a sphere from Theorem 4 and B_{n+2} is defined by (1), then the inversion in D_0 interchanges B_0 and B_{n+2} .

PROOF. The inversion in D_0 maps B_0 to $B_0 - 2(D_0 * B_0)D_0$. It is easy to verify that this is B_{n+2} .

COROLLARY 2. *Given a set $\{B_0, B_1, \dots, B_{n+1}\}$ of γ -clinal $(n-1)$ -spheres with $\gamma < -n^{-1}$, there is a set of $-(n + \gamma^{-1})^{-1}$ -clinal spheres $\{D_0, D_1, \dots, D_{n+1}\}$ such that each D_i is perpendicular to B_j for all $j \in \{0, 1, \dots, n+1\} \setminus \{i\}$.*

PROOF. Since $D_l = -a(n + \gamma^{-1})B_l + a\sum_{j=0, j \neq l}^{n+1} B_j$, for $l \neq k$,

$$D_l * D_k = D_l * aB_l = aB_l * D_l = a[-a(n + \gamma^{-1}) + a\gamma(n+1)] = -(n + \gamma^{-1})^{-1}.$$

This is an extension of Beecroft's result on tangent circles in the plane. Another proof due to Gerber can be found in [2].

COROLLARY 3. $B_0 * B_{n+2} = D_0 * D_{n+2} = 2\gamma(n+1)(n + \gamma^{-1})^{-1} - 1$.

4. Sequences of isoclinal spheres. In the last section we saw that any ordered set of $n+1$ γ -clinal spheres $\{B_1, \dots, B_{n+1}\}$ with $\gamma \neq -n^{-1}$, $-(n+1)^{-1}$ can be completed to a set of $n+2$ γ -clinal spheres in exactly two ways. Hence the spheres belong to just one sequence

$$(2) \quad \dots, B_{-1}, B_0, B_1, B_2, \dots, B_{n+1}, B_{n+2}, \dots$$

with the property that every $n+2$ consecutive members are γ -clinal. Hence we see that the relation (1) remains valid when all the indices are increased or decreased for any integer, so that it provides a recursion formula for computing coordinates of any sphere in the sequence (2), providing we know coordinates of $B_0, \dots, B_{n+1}$. Furthermore, formula (1) enables us to compute inclination between any two spheres in the sequence.

Everything we said so far was true for Euclidean, spherical and hyperbolic geometry since inversive geometry can be used to model any of them. For the details see [5]. In Euclidean space E^n given a sequence (2) we have

$$\beta_{n+2} = -\beta_0 + \frac{2}{n + \gamma^{-1}}(\beta_1 + \dots + \beta_{n+1}),$$

where β_i is the bend of B_i . The corresponding formula in spherical space S^n is

$$\cot \theta_{n+2} = -\cot \theta_0 + \frac{2}{n + \gamma^{-1}}(\cot \theta_1 + \dots + \cot \theta_{n+1}),$$

where θ_i is the angular radius of B_i . In hyperbolic space we have

$$\kappa_{n+2} = -\kappa_0 + \frac{2}{n + \gamma^{-1}}(\kappa_1 + \dots + \kappa_{n+1}),$$

where $\kappa_i = \coth \rho_i$ when the B_i are spheres of radius ρ_i , $\kappa_i = \tanh \mu_i$ where the B_i are equidistant surfaces with distance μ_i , and κ_i is unity when the B_i are horospheres. All of the formulas above are obtained from (1) by multiplying B_{n+2} with an appropriate vector as explained in [5].

5. Transformation mapping the sequence onto itself. As we already saw, the group $\mathfrak{M}_n$ generated by inversions is transitive on sets of $n+2$ γ -clinal spheres, so that there is exactly one transformation $M \in \mathfrak{M}_n$ mapping $B_0, \dots, B_{n+1}$ to $B_1, \dots, B_{n+2}$

successively and by (1)

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & \frac{2}{n + \gamma^{-1}} & \frac{2}{n + \gamma^{-1}} & \cdots & \frac{2}{n + \gamma^{-1}} \end{bmatrix}.$$

An inductive argument shows that M maps the whole sequence (2) onto itself by mapping any sphere in the sequence to its successor, i.e. $B_i M = B_{i+1}$ for all $i \in \mathbf{Z}$.

The characteristic polynomial of M is

$$\lambda^{n+2} - \frac{2}{n + \gamma^{-1}}(\lambda^{n+1} + \lambda^n + \cdots + \lambda) + 1.$$

The roots of the characteristic polynomial satisfy

$$(3) \quad \frac{\lambda^{n+2} - 1}{\lambda^{n+2} + 1} = (n + 1 + \gamma^{-1}) \frac{\lambda - 1}{\lambda + 1}.$$

Let us first investigate the complex roots of (3). The substitution $\lambda = e^{2i\theta}$ yields

$$(4) \quad \tan(n + 2)\theta = (n + 1 + \gamma^{-1})\tan \theta.$$

If n is even it is easy to see that there are exactly n complex roots of (3). If n is odd it is easy to see that there are $n - 1$ nonzero roots of (4). Furthermore, for $\gamma > -(n + 2)(n^2 + 3n + 1)^{-1}$,

$$\lim_{\theta \rightarrow \pi/2} \tan(n + 2)\theta / (n + 1 + \gamma^{-1})\tan \theta = (n + 1 + \gamma^{-1})^{-1}(n + 2)^{-1} > 1.$$

Hence there is $\theta \in ((n + 1)\pi/(n + 2), \pi/2)$ such that

$$\tan(n + 2)\theta > (n + 1 + \gamma^{-1})\tan \theta$$

and by the intermediate value theorem there are two more (conjugate) roots of (3).

Let $\gamma < -n^{-1}$. If there is a root $\lambda > 1$ of (3) it must be equal to $e^{2\delta}$ for some $\delta > 0$, and hence δ is a root of

$$(5) \quad \tanh(n + 2)\delta = (n + 1 + \gamma^{-1})\tanh \delta.$$

Since $\tanh(n + 2)\delta / \tanh \delta$ is a monotonic function of δ decreasing from $n + 2$ to 1 (see [1, p. 120]) and $1 < n + 1 + \gamma^{-1} < n + 1$, there is exactly one positive root of (5). Hence $e^{\pm 2\delta}$ are reciprocal real roots of (3).

Let $-n^{-1} < \gamma < -(n + 1)^{-1}$ and let n be even. If there is a root $\lambda < -1$ of (3) it must be equal to $-e^{2\delta}$ for some $\delta > 0$, and δ is a root of

$$(6) \quad \tanh(n + 2)\delta = (n + 1 + \gamma^{-1})\coth \delta.$$

By the intermediate value theorem it is enough to show that there is a δ such that $\tanh(n+2)\delta > (n+1+\gamma^{-1})\coth\delta$. This is true since

$$\lim_{\delta \rightarrow \infty} \tanh(n+2)\delta / (n+1+\gamma^{-1})\coth\delta = (n+1+\gamma^{-1})^{-1} > 1.$$

Let $-\gamma^{-1} < \gamma < -(n+2)(n^2+3n+1)^{-1}$ and $\lambda = -e^{2\delta}$ for some $\delta > 0$, so that

$$(7) \quad \tanh(n+2)\delta = (n+1+\gamma^{-1})^{-1} \tanh\delta.$$

Since $1 < (n+1+\gamma^{-1})^{-1} < n+2$ there is exactly one positive root of (7) and hence two negative reciprocal roots $-e^{\pm 2\delta}$ of (3).

Finally if n is odd, -1 is a root of (3). If furthermore, $\gamma = -(n+2)(n^2+3n+1)^{-1}$ then -1 is a triple root of (3).

Hence for every possible γ and n we found $n+2$ roots of (3), i.e. all fixed elements of M . Each real solution $\lambda \neq \pm 1$ yields a fixed vector A representing a fixed point of M since $A * A = AM * AM = \lambda^2 A * A$ and hence $A * A = 0$. For $\lambda = -1$ the corresponding eigenvector is

$$B = B_0 - \left(1 + \frac{2}{n+\gamma^{-1}}\right) B_1 + B_2 - \left(1 + \frac{2}{n+\gamma^{-1}}\right) B_3 \\ + \cdots - \left(1 + \frac{2}{n+\gamma^{-1}}\right) B_n + B_{n+1}.$$

Lengthy calculation shows that $B * B \neq 0$ and hence when n is odd, transformation M has the fixed sphere B .

We conclude that when n is even, M has two fixed points and the only Möbius transformations with two fixed points are dilative rotations (see [6]). When n is odd and $\gamma < -(n+2)(n^2+3n+1)^{-1}$ we have two fixed points lying on the fixed sphere (since $A * B = AM * BM = -\lambda A * B$, i.e. $A * B = 0$), and M is a dilative rotatory reflection; finally for n odd and $-(n+2)(n^2+3n+1)^{-1} \leq \gamma < -(n+1)^{-1}$ we have one fixed sphere and M is a rotatory reflection. All the angles of rotation are given as solutions of the equation (4). Coefficients of dilation for the dilative rotation and the dilative rotatory reflection are solutions of (5), (6) or (7) depending on γ and n as indicated above.

6. Loxodromic sequences.

Let us consider the inclination between the fixed sphere B and any B_i in the sequence (2) $B * B_i = BM * B_i M = -B * B_{i+1}$. Hence all the spheres in the sequence have the same inclination with the fixed sphere and their centres are alternately on the different sides of the fixed sphere. Furthermore

$$B * B_0 = 1 - (n+1+\gamma^{-1})(n+\gamma^{-1})^{-1}\gamma(n+1),$$

so that $B * B_0$ is > -1 for $\gamma < -n^{-1}$ and < -1 otherwise, so that for $\gamma < -n^{-1}$ and are disjoint from it.

For even n (using complex coordinates) we can write the dilative rotation as $(z_1, \dots, z_{n/2}) \rightarrow e^{2\delta}(z_1 e^{2i\theta_1}, \dots, z_{n/2} e^{2i\theta_{n/2}})$ (see [6]). When $\gamma = -1$, so that $n+1$

consecutive spheres are tangent, choose any point of contact A between two spheres; the set of points $\{AM^i\}_{i \in \mathbb{Z}}$ lies on the curve

$$(8) \quad \gamma(t) = e^{2\delta t} (a_1 e^{2i\theta_1 t}, \dots, a_{n/2} e^{2i\theta_{n/2} t})$$

(see [4]). Now if A_0 is any point on B_0 and $A_i = A_0 M^i$, $i \in \mathbb{Z}$, then A_i is a point on B_i since $0 = A_0 * B_0 = A_0 M^i * B_0 M^i = A_i * B_i$, and the set of points $\{A_i\}_{i \in \mathbb{Z}}$ lies on a curve from the family (8) for a proper choice of $a_1, \dots, a_{n/2}$. In odd dimensions dilative rotatory reflection restricts on the fixed sphere B to dilative rotation and if A_0 is a point on $B_0 \cap B$ (this is possible only if $\gamma < -n^{-1}$) then $\{A_i\}_{i \in \mathbb{Z}}$ are again points on a curve from the family (8) which is contained in B . The fixed points of M are the accumulation points of the curve.

FIGURE 1

In particular when $n = 2$ the curves are loxodromes. Clearly a loxodrome intersects all circles at the same angle. For $\gamma < -1$ this angle varies continuously from 0 to π as shown on Figure 1, and for a particular choice of a starting point it will be $\frac{\pi}{2}$. This loxodrome corresponds to the loxodrome described above for the case $\gamma = -1$. For $n = 3$ and $\gamma < -n^{-1}$ the curves are loxodromes lying on the fixed sphere.

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