ON THE NONEXISTENCE OF RIEMANNIAN SUBMERSIONS FROM $\mathbb{CP}(7)$ AND $\mathbb{QP}(3)$

JACK UCCI

ABSTRACT. We show the nonexistence of a Serre fibration of the form $\mathbb{CP}(3) \rightarrow \mathbb{CP}(7) \rightarrow S^8$.

Introduction. A Riemannian submersion $\pi: M \rightarrow B$ between Riemannian manifolds is a C^∞ -mapping π of maximal rank such that its differential π_* preserves the length of vectors orthogonal to fibers. When M is a complex projective r -space $\mathbb{CP}(r)$ and all fibers of π are complex, connected and totally geodesic with $2 \leq \dim \text{fiber} \leq 2r - 2$, Richard Escobales [4] has shown that π must belong to one of the following two classes: (i) $S^2 \rightarrow \mathbb{CP}(2n+1) \xrightarrow{\pi} \mathbb{QP}(n)$, (ii) $\mathbb{CP}(3) \rightarrow \mathbb{CP}(7) \xrightarrow{\pi} S^{8(\frac{1}{2})}$ ($S^{8(\frac{1}{2})}$ denotes the 8-sphere of radius $\frac{1}{2}$). It is a classical fact that class (i) is nonempty, and Escobales has asked if class (ii) is empty. Our result implies that this is indeed the case. We show, more generally,

THEOREM. *There exists no Serre fibration of the form $\mathbb{CP}(3) \rightarrow \mathbb{CP}(7) \rightarrow S^8$.*

Our original result [8] considered only the case of fibre bundles and the above extension answers a question posed by the referee. The proof, however, remains essentially unchanged. Reinhard Schultz has also shown me how the extended result follows from his work [6] and the existence of a transfer map [1].

I wish to thank Richard Escobales for his enthusiastic proposal of this question, Phil Church for discussions concerning the argument in §2 (and for suggesting references [2, 9]), the editor Reinhard Schultz and Bob Stong—each of the latter two noted the relevance of [6] and the prime $p = 3$.

1. The composition $S^{2r+1} \rightarrow \mathbb{CP}(r) \rightarrow S^{r+1}$. Let S^1 act in the standard way on S^{2r+1} with quotient CW complex $\mathbb{CP}(r) = S^{2r+1}/S^1$. A mapping $f: S^{2r+1} \rightarrow X$ is called *strictly S^1 -invariant* if $f = g \circ \pi$ for some mapping $g: \mathbb{CP}(r) \rightarrow X$. A homotopy class $\alpha \in \pi_{2r+1}(X)$ is called *S^1 -invariant* if it possesses a strictly S^1 -invariant representative.

PROOF OF THEOREM. If $\mathbb{CP}(3) \rightarrow \mathbb{CP}(7) \xrightarrow{g} S^8$ were a Serre fibration, the composite $S^1 \rightarrow S^{15} \xrightarrow{f} S^8$, $f = g \circ \pi$, would also be a Serre fibration, where $S^1 \rightarrow S^{15} \xrightarrow{\pi} \mathbb{CP}(7)$

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is the (differentiable) fibre bundle defined by the standard S^1 -action on S^{15} . A portion of the exact homotopy sequence for f is given by

$$\begin{array}{ccccccc} \pi_{15}S^7 & \rightarrow & \pi_{15}S^{15} & \xrightarrow{f_{\#}} & \pi_{15}S^8 & \rightarrow & \pi_{14}S^7 \\ \wr & & \wr & & \wr & & \wr \\ \text{finite} & & Z & & Z + Z_{120} & & Z_{120} \end{array}$$

So $f_{\#}\iota = [\iota]$ must be of the form $(\pm 1, k)$, where $(1, 0)$ denotes some generator of the infinite cyclic summand and $(0, 1)$ denotes an element of order 120. Otherwise we would have order $\pi_{14}S^7 > 120$. Hence the Hopf invariant $H(f)$ of f is ± 1 .

We now have a strictly S^1 -invariant mapping f of Hopf invariant ± 1 . The following (suitably modified) argument of Randall [5] shows that such a map cannot exist. We have a diagram of commutative squares and horizontal cofibrations

$$\begin{array}{ccccc} S^{15} & \xrightarrow{\pi} & \text{CP}(7) & \rightarrow & \text{CP}(8) \\ \parallel & & \downarrow g & & \downarrow G \\ S^{15} & \xrightarrow{f} & S^8 & \rightarrow & S^8 \cup_f e^{16} \end{array}$$

Because $H(f) = \pm 1$, $u^2 = v$ for suitably chosen generators $u \in H^8(S^8 \cup_f e^{16}; Z)$ and $v \in H^{16}(S^8 \cup_f e^{16}; Z)$. Now $G^*v = x^8$ where $x \in H^2(\text{CP}(8); Z)$ is a generator. We claim $G^*u = 3kx^4$ for some integer k . For this note that for $\text{CP}(r)$, $r \geq 3$, $P^1 r_3(x) = r_3 x^3$ where P^1 is the 1st mod 3 Steenrod cohomology operation and r_3 is reduction mod 3. Since P^1 acts trivially in $S^8 \cup_f e^{16}$, $G^*r_3 u = g^*r_3 u = 0$. This establishes the claim that $G^*u = 3kx^4$.

But we now have $x^8 = G^*u^2 = (G^*u)^2 = 9k^2 x^8$, which is impossible. Hence no Serre fibration of the form $\text{CP}(3) \rightarrow \text{CP}(7) \rightarrow S^8$ can exist.

COROLLARY. *There exists no Serre fibration of the form $S^4 \rightarrow \text{QP}(3) \xrightarrow{h} S^8$.*

Otherwise the composition $\text{CP}(7) \xrightarrow{\pi'} \text{QP}(3) \xrightarrow{h} S^8$ would be a Serre fibration, contradicting our result.

The preceding argument establishes the following more general result.

PROPOSITION. *If $\alpha \in \pi_{4n-1}S^{2n}$ is an S^1 -invariant class and p is any prime with $p \nmid n$ and $p < n + 1$, then p^2 divides the Hopf invariant $H(\alpha)$ of α .*

2. Concluding remarks. As is well known, Riemannian submersions under mild assumptions (which obtain in our situation) are fibre bundle projections. So our original approach considered the question of when the composition

$$S^{2r+1} \xrightarrow{\pi} \text{CP}(r) \xrightarrow{g} S^{r+1}$$

is the projection of a fibre bundle. In the differentiable case, we have

PROPOSITION. *Let $f: S^{2r+1} \rightarrow S^{r+1}$ be a strictly S^1 -invariant mapping. Then f is a differentiable fibre bundle if and only if the induced mapping $g: \text{CP}(r) \rightarrow S^{r+1}$ is a differentiable fibre bundle.*

PROOF. First recall a result of J. A. Wolf [9] which asserts that a differentiable proper mapping, $\phi: E \rightarrow B$, of rank $\dim B$ from a connected, paracompact space to a paracompact space is a locally trivial differentiable fibre space, i.e. a differentiable fibre bundle.

Now if g is a differentiable fibre bundle, then f and g have maximal rank, since π has. Then f is proper, since both S^{2r+1} and S^{r+1} are compact and Hausdorff and so f is a differentiable fibre bundle.

Conversely, f and π have maximal rank and so by the Implicit Function Theorem [7, p. 14 of Appendix 1] f and π are locally projection mappings of the form $h: R^{n+k} \rightarrow R^n$. However, the factorization $f = g \circ \pi$ implies that g also is locally a projection mapping and so is differentiable of maximal rank. But g is proper, hence a differentiable fibre bundle.

The preceding proposition (and argument) is true in the topological case if we replace Wolf's result by that of Cheeger and Kister (implicit in [2]) which asserts that a continuous proper map $\phi: E \rightarrow B$ of topological manifolds with B connected, which is locally topologically equivalent to a projection map, is a topological fibre bundle.

Finally we include the proof of the Theorem due to R. Schultz. From [1] a fibration $CP(3) \rightarrow CP(7) \xrightarrow{f} S^8$ would give an S -map $\tau: S^8 \rightarrow CP(7)$ such that $\tau^*f^* =$ multiplication by the Euler characteristic $\chi(CP(3)) = 4$ on $H^*(S^8; R) - H^*(; R)$ denotes singular cohomology with coefficients in any ring R . When $R = \mathbb{Z}/3$, $H^*(CP(7); \mathbb{Z}/3)$ splits over the mod 3 Steenrod algebra (as in [6]) with one of the summands isomorphic to $H^*(S^8; \mathbb{Z}/3)$. But in $H^8(CP^7; \mathbb{Z}/3)$ we know $P^1x^2 = 2x^4 \neq 0$ and so such a splitting is impossible.

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DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13210