THE NONTRIVIALITY OF THE FIRST RATIONAL HOMOLOGY GROUP OF SOME CONNECTED INVARIANT SUBSETS OF PERIODIC TRANSFORMATIONS

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ABSTRACT. This note was inspired by some results of P. A. Smith [S]. One proves that for any periodic map of a manifold M and any codimension two invariant submanifold P of M containing part of the stationary point set, connected invariant subsets of the complement of P must carry nontrivial one-dimensional rational cycles, provided that M satisfies some simple homological conditions (Theorem A). This fact has interesting consequences in transformation group theory.

0. Introduction. If $m \geq 2$ is a positive integer let $Z_m = \mathbf{Z}_m$ be the cyclic group of order m and if $m = \infty$ let Z_m resp. $\mathbf{Z}_m$ be the infinite cyclic group, respectively, $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. We also denote by $G_{m,n}$, $m, n = 2, 3, 4, \dots, \infty$, a semidirect product $\mathbf{Z}_m \rtimes_t \mathbf{Z}_n$ for $t: \mathbf{Z}_n \rightarrow \text{Aut } \mathbf{Z}_m$. Such a semidirect product has the inclusion $\mathbf{Z}_m \xrightarrow{i} G_{m,n}$, the projection $G_{m,n} \xrightarrow{\pi} \mathbf{Z}_n$ and the section $s: \mathbf{Z}_n \rightarrow G_{m,n}$ ($\pi \circ s = \text{id}$) as part of the data. Clearly if $n = \infty$, $G_{m,n} = \mathbf{Z}_m \times S^1$. The groups $G_{m,n}$ are regarded as compact Lie groups.

Given a compact Lie group G , $\mu: G \times M \rightarrow M$ a topological action, N an invariant submanifold, and $x \in M$, then the action $\mu: G \times (M, N) \rightarrow (M, N)$ is called locally smooth at x if there exists a smooth action $\tilde{\mu}: G \times V \rightarrow V$, an invariant submanifold $W \subset V$, and an invariant neighborhood $\mathcal{U}$ of $x \in M$ together with an equivariant homeomorphism $\psi(\mathcal{U}, \mathcal{U} \cap N) \rightarrow (V, W)$. The main result of this note is the following:

THEOREM A. *Let $\mu: \mathbf{Z}_m \times M^n \rightarrow M^n$ be a smooth (topological) orientation preserving action, $P^{n-2} \subset M^n$ an invariant smooth (locally flat) closed oriented submanifold whose orientation is also invariant. Assume that $H_1(M)$ is torsion prime to m , $H_2(M; \mathbf{Q}) = 0$, and*

- (i) $M^{\mathbf{Z}_m} \neq \emptyset^2$ and $P \cap M^{\mathbf{Z}_m} \neq \emptyset$ is a union of connected components of $M^{\mathbf{Z}_m}$,
- (ii) if μ is a topological action then there exists $y \in P \cap M^{\mathbf{Z}_m}$ so that $\mu: \mathbf{Z}_m \times (M, P) \rightarrow (M, P)$ is locally smooth at y .

Received by the editors July 20, 1982.

1980 *Mathematics Subject Classification.* Primary 57S10, 57S15, 57S17.

¹Both authors were supported by NSF grants. The authors are grateful to R. Schultz and R. Stong for helpful suggestions.

²For an action $\mu: G \times M \rightarrow M$ one denotes by M^G the fixed point set of μ .

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0002-9939/82/0000-1349/\$02.75

Then

(a) for any connected invariant subset $X \subseteq M \setminus P$, $H_1(X; \mathbb{Q}) \rightarrow H_1(M \setminus P; \mathbb{Q})$ and $H_1(X; \mathbb{Z}_m) \rightarrow H_1(M \setminus P; \mathbb{Z}_m)$ are nontrivial,

(b) $P \supset M^{\mathbb{Z}_m}$.

Theorem A is applied to transformation groups through

COROLLARY A'. Let M, μ, P be as in Proposition A and let $K^r \subset M^n$ be a nonempty connected invariant submanifold with $H_1(K^r; k) = 0$ for $k = \mathbb{Q}$ or $\mathbb{Z}_m$. Then $K^r \cap P \neq \emptyset$ and $\dim(K^r \cap P) \geq r - 2$.

Corollary A' implies Theorem B and is an important tool in the proof of Theorem C.

THEOREM B (P. A. SMITH). Suppose $G_{p,q}$ acts locally smoothly and effectively on the acyclic topological manifold M^n preserving the orientation. Let p, q be distinct primes or ∞ , and $\dim(M^{\mathbb{Z}_q}) = n - 2$. Then $\dim M^{G_{p,q}} \neq \emptyset$ and $\dim M^{G_{p,q}} = (\dim M^{\mathbb{Z}_p}) - 2$.

Theorem B is due to P. A. Smith whose paper [S] has inspired this note. Smith has stated it for $G_{p,q} = \mathbb{Z}_{pq}$ and without any conclusion about the dimension $M^{G_{p,q}}$. Notice that if we drop the condition $\dim(M^{\mathbb{Z}_p}) = n - 2$ the conclusion is false (see Kister [K2]). A systematic class of fixed point free $G_{p,q}$ -actions has been constructed by Assadi [A].

A manifold M^{n+r} is called r -spheric if its universal cover has the homotopy type of S^r .

THEOREM C. (1) Let M^{n+2} be a 2-spheric topological manifold and $\mu: \mathbb{Z}_m \times M^{n+2} \rightarrow M^{n+2}$ be a locally smooth effective action with m prime or ∞ . If $M^{\mathbb{Z}_m} \neq \emptyset$ then

$$H_*(M^{\mathbb{Z}_m}; \mathbb{Z}_m) = H_*(S^0 \times K(\pi_1(M), 1); \mathbb{Z}_m)$$

and $\dim M^{\mathbb{Z}_m} = n$; hence, if $m = \infty$, $\pi_1(M)$ is a Poincaré duality group of formal dimension n .

(2) Let M^{n+2} be a 2-spheric topological manifold and $\mu: G_{m,r} \times M^{n+2} \rightarrow M^{n+2}$ an orientation preserving effective locally smooth action with $M^{\mathbb{Z}_m} \neq \emptyset$ and $M^{\mathbb{Z}_r} \neq \emptyset$. Then m and r are finite and $M^{\mathbb{Z}_m} \cap M^{\mathbb{Z}_r}$ is a nonempty manifold with all connected components of dimension n and $n - 2$. If there are connected components of dimension n then $G_{m,r} = \mathbb{Z}_{mr}$. If all components have dimension n then $M^{\mathbb{Z}_m} = M^{\mathbb{Z}_r} = M^{G_{m,r}}$ and the action is semifree. If $m = r$ all components have dimension n .

The proof of Theorem A and Corollary A' will be given in §1, and the proof of Theorems B and C in §2.

1.

PROOF OF THEOREM A. It is easy to see that (a) implies (b) because any stationary point outside P forms a connected invariant subset with trivial homology. To prove (a) we observe that one can choose a closed tubular neighborhood of P , $\tilde{P}: \tilde{P} \rightarrow P$, $\tilde{P} \subset M$ so that for at least one $y \in P \cap M^{\mathbb{Z}_m}$, $\tilde{\pi}^{-1}(y)$ is invariant. If the action is smooth, a closed invariant tubular neighborhood is the right choice for $\tilde{P}$. If the action is not smooth, let $y \in P \cap M^{\mathbb{Z}_m}$ so that $\mu: \mathbb{Z}_m \times (M, P) \rightarrow (M, P)$ is locally

smooth at y . We choose an invariant tubular neighborhood with fibre discs for a small neighborhood of y in P . Eventually shrinking this neighborhood, we can extend this to a closed tubular neighborhood of P because, for locally flat P , this is possible by a theorem of Kirby [K1] which states the existence and uniqueness of the closed tubular neighborhood for closed locally flat embeddings in codimension 2. Let $E = \partial\tilde{P}$ and $\pi = \tilde{\pi}|_{\partial\tilde{P}}: \pi: E \rightarrow P$ being the restriction of $\tilde{\pi}$ to $\partial\tilde{P} = E$ is a bundle with fibre S^1 . Let us denote by $\tilde{E}$ the complement of $\text{Int}(\tilde{P})$ in M so that $E = \tilde{E} \cap \tilde{P}$. The first step in our proof is to show that $H_1(M \setminus P) = H_1(\tilde{E}; Z)$ is isomorphic to $Z \oplus G$ where G is a torsion group with $G \otimes Z_m = 0$, and for any $z \in P$ the map $i_z: S^1 = \pi^{-1}(z) \rightarrow E \rightarrow M \setminus P$ represents in the homology group $H_1(M \setminus P; Z)$ an element of the form $(t, g) \in Z \oplus G$ with t prime to m .

Assuming the first step proved, the proof of Theorem A(a), for $m = \infty$, goes as follows: Let $\mu_x: S^1 \rightarrow M \setminus P$ be the restriction of the action μ to $S^1 \times x$ where $x \in M \setminus P$. Since any connected invariant subset of $M \setminus P$, say X , contains $\mu_{x_0}(S^1)$ for $x_0 \in X$, it suffices to show that for each x , $\mu_x: S^1 \rightarrow M \setminus P$ induces a nontrivial homomorphism $(\mu_x)^*: H_1(S^1; Q) \rightarrow H_1(M \setminus P; Q)$. Because $M \setminus P$ is connected it suffices to verify this for one x since if $x', x'' \in M \setminus P$, $\mu_{x'}$ is homotopic to $\mu_{x''}$. Now choose $x \in \pi^{-1}(y)$ where $y \in P \cap M^{S^1}$ is a point where $\tilde{\pi}^{-1}(y)$ is S^1 -invariant; $\mu_x: S^1 \rightarrow \mu_x(S^1) = \pi^{-1}(y)$ is a finite cover. Therefore the image of the generator $u \in H_1(S^1; Z)$ in $H_1(M \setminus P; Z)$ is (kt, kg) with $k \neq 0$. This proves our statement for $m = \infty$.

If $m \neq \infty$ the proof requires supplementary considerations. Let $\mu: \mathbf{Z}_m \times M \rightarrow M$ be our action and let $\sigma: M \rightarrow M$ the homeomorphism induced by the generator. Given an invariant subset $U \subset M$, a 1-dimensional singular simplex $l: [0, 1] \rightarrow U$ is called special if $l(1) = \sigma(l(0))$. Each special simplex l defines a map $\mu^l: S^1 \rightarrow U$ by taking

$$\mu^l(e^{i\theta}) = \begin{cases} l\left(\frac{\theta m}{2\pi}\right) & \text{if } 0 \leq \theta \leq \frac{2\pi}{m}, \\ \sigma \circ l\left(\frac{\theta m - 2\pi}{2\pi}\right) & \text{if } \frac{2\pi}{m} \leq \theta \leq \frac{4\pi}{m}, \\ \sigma^{m-1} \circ l\left(\frac{\theta m - 2\pi(m-1)}{2\pi}\right) & \text{if } \frac{2\pi}{m}(m-1) \leq \theta \leq 2\pi, \end{cases}$$

which represents the integral cycle $\Sigma l = l + \sigma l + \cdots + \sigma^{m-1}l$. If $l, l': [0, 1] \rightarrow U$ are two special simplices let $\gamma: [0, 1] \rightarrow U$ be a singular 1-dimensional simplex with $\gamma(0) = l(0)$, $\gamma(1) = l'(0)$. The 1-dimensional singular chain $\Delta = \gamma + l' - \sigma\gamma - l$ is obviously a cycle and $\Sigma l = \Sigma l' - \Delta - \sigma\Delta - \cdots - \sigma^{m-1}\Delta$. In the sequel, let U be $M \setminus P$. As a second step of our proof we will show that: if σ_* is the isomorphism of $H_1(M \setminus P; Z) = Z \oplus G$, resp. of $H_1(M \setminus P; Z_m) = Z_m$ induced by σ , then $\sigma_*(s, g) = (s, \theta(s, g))$ where $s \in Z$, $g, \theta(s, g) \in G$, resp. $\sigma_*(s) = s$, $s \in Z_m$. Assuming this second step proved, it suffices to show that for some special 1-simplex l , the integral homology class of $[\Sigma l]$ resp. the homology class mod m , $[\Sigma l]_m$ is of the form $(t', g') \in Z \oplus G$ with t' prime to m resp. $t' \in Z_m$ where $t' \neq 0$. Then we can conclude that for any other special simplex l' , $[\Sigma l'] = [\Sigma l] + [\Delta] + \sigma_*[\Delta] + \cdots + \sigma_*^{m-1}[\Delta] = (t' + mr, g'') \in Z \oplus G$ where $[\Delta] = (r, g'')$ resp. $[\Sigma l']_m = [\Sigma l]_m = t'$. Hence any integral homology class $[\Sigma l']$ is of infinite order and nontrivial

resp. any mod m homology class $[\Sigma l'] = \mathbf{t}'$ is hence nontrivial. Now let us take l to be a special cycle lying on $\pi^{-1}(y)$, $y \in P \cap M^{\mathbb{Z}_m}$ where $\tilde{\pi}^{-1}(y)$ is invariant. We can choose l so that $[\Sigma l] = c[\pi^{-1}(y)]$ where $0 < c < m$ and c divides m . Hence $[\Sigma l] = [ct, g']$ with $ct \neq 0 \bmod m$; thus $[\Sigma l]_m \neq 0$ which verifies the claim of Theorem A(a).

To prove Step 1, consider the exact sequence

$$(*) \quad \rightarrow H_2(\tilde{E}; k) \xrightarrow{i_2} H_2(M; k) \xrightarrow{j_2} H_2(M, \tilde{E}; k) \rightarrow H_1(\tilde{E}; k) \xrightarrow{i_1} H_1(M; k) \rightarrow \dots$$

associated with the pair $(M, \tilde{E})$ and observe that because $H_2(M; \mathbb{Q}) = 0$ and

$$H_i(M, \tilde{E}; k) = H_i(\tilde{P}, E; k) = \begin{cases} 0 & \text{if } i = 0, 1, \\ k & \text{if } i = 2, \end{cases}$$

j_2 is zero. Hence $(*)$ shortens to

$$0 \rightarrow k \rightarrow H_1(\tilde{E}; k) \rightarrow H_1(M; k) \quad \text{and} \quad H_2(\tilde{E}; k) \xrightarrow{i_2} H_2(M; k) \rightarrow 0.$$

By hypotheses $H_1(M; \mathbb{Z})$ is torsion prime to m and for $k = \mathbb{Z}_m$, $H_1(\tilde{E}; \mathbb{Z}_m) = \mathbb{Z}_m$. This implies $H_1(\tilde{E}; \mathbb{Z}) = \mathbb{Z} \oplus G$ with G a torsion group so that $G \otimes \mathbb{Z}_m = 0$.

The Mayer-Vietoris sequence applied to $M = \tilde{E} \cup \tilde{P}$, $E = \tilde{E} \cap \tilde{P}$ gives the exact sequence

$$(**) \quad \rightarrow H_2(\tilde{E}; k) \oplus H_2(\tilde{P}; k) \xrightarrow{i_2 \oplus \pi_2} H_2(M; k) \xrightarrow{\delta} H_1(E; k) \\ \xrightarrow{i \oplus \pi_1} H_1(\tilde{E}; k) \oplus H_1(\tilde{P}; k) \rightarrow H_1(M; k).$$

Because i_2 is surjective, so is $i_2 \oplus \pi_2$, hence δ is zero. Thus $(**)$ shortens to

$$0 \rightarrow H_1(E; k) \xrightarrow{i \oplus \pi_1} H_1(\tilde{E}; k) \oplus H_1(\tilde{P}; k) \rightarrow H_1(M; k).$$

Since in the diagram below the horizontal line is the abelianization of the homotopy exact sequence of the bundle $S_y^1 \xrightarrow{i_y} E \rightarrow P$

$$\begin{array}{ccccc} H_1(S_y^1; k) & \xrightarrow{(i_y)_1} & H_1(E; k) & \xrightarrow{\pi_1} & H_1(P; k) \rightarrow 0 \\ \uparrow \cong & & \nearrow & & \\ H_2(\tilde{\pi}^{-1}(y), \pi^{-1}(y); k) & & & & \\ \downarrow \cong & & \nearrow & & \\ H_2(\tilde{P}, E; k) & & & & \end{array}$$

and δ^E is injective, $(i_y)_1$ has to be injective. δ^E is injective because in the commutative diagram below $\Delta^{\tilde{E}}$ is injective and w is an isomorphism:

$$\begin{array}{ccc} H_2(M, \tilde{E}; k) & \xrightarrow{\delta^{\tilde{E}}} & H_1(\tilde{E}; k) \\ w \uparrow \cong & & \uparrow \\ H_2(\tilde{P}, E; k) & \xrightarrow{\delta^E} & H_1(E; k) \end{array}$$

Consequently we have the exact sequence

$$0 \rightarrow H_1(S_y^1; k) \rightarrow H_1(E; k) \rightarrow H_1(P; k) \rightarrow 0.$$

Since for $k = \mathbb{Z}_m$, $i_1 \oplus \pi_1$ is an isomorphism, the composition $H_1(S_y^1; \mathbb{Z}_m) \rightarrow H_1(E; \mathbb{Z}_m) \rightarrow H_1(\tilde{E}; \mathbb{Z}_m)$ is an isomorphism, hence $t \equiv 1 \pmod{m}$ and the first step is proved.

To prove Step 2 we observe that by Step 1 the component Z of $H_1(\tilde{E}; \mathbb{Z}) = H_1(M \setminus P; \mathbb{Z})$ comes from $H_2(M, \tilde{E}; \mathbb{Z}) = H_2(M, M \setminus P; \mathbb{Z})$ which is invariant under the action of $\mathbb{Z}_m$ since $H_2(M, \tilde{E}; \mathbb{Z}) \xrightarrow{\sim} H_2(\tilde{\pi}^{-1}(y), \pi^{-1}(y); \mathbb{Z})$ and the action preserves the orientations of M and P ; so that it is trivial on $H_2(\tilde{\pi}^{-1}(y), \pi^{-1}(y); \mathbb{Z})$.

PROOF OF COROLLARY A'. If $K' \cap P = \emptyset$ or $\dim K' \cap P < n - 2$, then $K \setminus K \cap P$ will be connected and provides an example of an invariant set with

$$H_1(K \setminus K \cap P; \mathbb{Q}) = 0,$$

which is impossible by Proposition A.

The following example (due to R. Stong) shows that condition (i) of Theorem A cannot be weakened to $P^{\mathbb{Z}_m} \neq \emptyset$.

EXAMPLE. Consider $M^n = S^{2j+1}$ with $\mathbb{Z}_m$ action $\rho(z_0, z_1, \dots, z_j) = (z_0, z_1, e^{2\pi i z_2/m}, \dots, e^{2\pi i z_j/m})$ and let $P^{n-2} = S^{2j-1} = \{z | z_0 = 0\}$, where $n = 2j + 1$ and $j \geq 2$. Clearly M and P are connected, $H_2(M, \mathbb{Q}) = H_1(M, \mathbb{Z}) = 0$, $P^{\mathbb{Z}_m} \neq \emptyset$ and $M^{\mathbb{Z}_m} \not\subset P$. Also every subset of $S^1 = \{z | z_i = 0 \text{ for } i > 0\}$ is invariant and need not have a nontrivial image in $H_1(M - P; \mathbb{Z}_m)$. In fact, there are invariant circles which are homologically trivial.

2.

PROOF OF THEOREM B. Let $P = M^{\mathbb{Z}_q}$ and $K = M^{\mathbb{Z}_p}$, and take $\mu: Z_q \times M \rightarrow M$ to be the induced action by a subgroup isomorphic to Z_q . Then K is invariant under the Z_q -action and $K - M^{G_{p,q}} \subset M - P$. By Smith theory $\bar{H}_*(K; \mathbb{Z}_p) = 0 = \bar{H}_*(P; \mathbb{Z}_q)$. Thus K and P are both connected. If $p = \infty$, then $\bar{H}_*(K; \mathbb{Q}) = 0$ and the conclusion follows from Corollary A'. If $p < \infty$ and $\dim M^{G_{p,q}} < \dim M^{\mathbb{Z}_p} - 2$ (in particular if $M^{G_{p,q}} = \emptyset$), then $H_1(K - M^{G_{p,q}})$ will be p -divisible. The linking number of any 1-cycle of $M - P$ with P is well defined since $H_1(M) = H_2(M) = 0$.

If we choose $\tilde{\pi}: \tilde{P} \rightarrow P$ and $y \in P$ as in the proof of Theorem A, it is easily seen that one can find special 1-cycles in $\tilde{\pi}^{-1}(y) - \{y\} \subset M - P$ whose linking number with P is one. However, if a special 1-cycle lies on $K \setminus M^{G_{p,q}}$, its linking number with P must vanish, being p -divisible. This contradiction establishes Theorem B.

PROOF OF THEOREM C. Let $\tilde{M} \xrightarrow{f} M$ be the universal covering of M . Since $M^{\mathbb{Z}_m} \neq \emptyset$, the action μ can be lifted to an action $\tilde{\mu}: \mathbb{Z}_m \times \tilde{M} \rightarrow \tilde{M}$ with $f^{-1}(M^{\mathbb{Z}_m}) = \tilde{M}^{\mathbb{Z}_m}$; hence $\tilde{M}^{\mathbb{Z}_m} \xrightarrow{f} M^{\mathbb{Z}_m}$ is a principal covering with group $\pi_1(M) = \pi$. If m is a prime or ∞ , by Smith theory $\tilde{H}_*(M^{\mathbb{Z}_m}; \mathbb{Z}_m) = H_*(S^k; \mathbb{Z}_m)$ where $k = 0$ or 2 ; if $k = 2$ Smith theory equally implies that $H_*(\tilde{M}^{\mathbb{Z}_m}; \mathbb{Z}_m) \rightarrow H(M; \mathbb{Z}_m)$ is an isomorphism [B, Chapter III] which implies (comparing the spectral sequences associated with the fibrations $\tilde{M} \rightarrow M \rightarrow K(\pi, 1)$ and $\tilde{M}^{\mathbb{Z}_m} \rightarrow M^{\mathbb{Z}_m} \rightarrow K(\pi, 1)$) that M and $M^{\mathbb{Z}_m}$ have the same homology with coefficients in $\mathbb{Z}_m$; this is impossible, hence $k = 0$ and $f^{-1}(M^{\mathbb{Z}_m})$ consists of two connected components $\tilde{P}_1$ and $\tilde{P}_2$ each of them with trivial

homology mod m and being the total space of principal coverings with group π . By spectral sequence arguments applied to the fibrations $\tilde{P}_i \rightarrow K(\pi, 1)$ we have $H_*(P_i; Z_m) = H_*(K(\pi, 1); Z_m)$. We also claim that $\dim P_i = n$ because the composition $P_i \rightarrow M \rightarrow K(\pi, 1)$ induces an isomorphism for homology with coefficients in Z_m which combined with the Gysin sequence of the fibration $\tilde{M} \rightarrow M \rightarrow K(\pi, 1)$ implies $H_n(K(\pi, 1); Z_m) = H_n(P_i; Z_m) = Z_m$. This proves Theorem C(1).

Before we start the proof of (2) we notice that $\tilde{M} \setminus \tilde{P}_i$ satisfies $H_1(\tilde{M} \setminus \tilde{P}_i; Z_m) = 0$ and $H_2(\tilde{M} \setminus \tilde{P}_i; Q) = 0$ for m prime and finite. To see this, one considers the exact sequence of the pair $(\tilde{M}, \tilde{M} \setminus \tilde{P}_i)$,

$$\begin{aligned} \rightarrow H_3(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) \rightarrow H_2(\tilde{M} \setminus \tilde{P}_i; Z_m) \rightarrow H_2(\tilde{M}; Z_m) \\ \xrightarrow{\psi} H_2(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) \rightarrow H_1(\tilde{M} \setminus \tilde{P}_i; Z_m) \rightarrow 0 \end{aligned}$$

which, after applying the isomorphism $H_3(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) = H_1(\tilde{P}_i; Z_m)$ and Smith theory, $H_1(\tilde{P}_i; Z_m) = 0$ becomes

$$0 \rightarrow H_2(\tilde{M} \setminus \tilde{P}_i; Z_m) \rightarrow Z_m \xrightarrow{\psi} Z_m \rightarrow H_1(\tilde{M} \setminus \tilde{P}_i; Z_m) \rightarrow 0.$$

If m is prime (and finite) ψ is not zero, hence it is an isomorphism. To see this, assume $\psi = 0$; hence $Z_m = H_2(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) \rightarrow H_1(\tilde{M} \setminus \tilde{P}_i; Z_m)$ is an isomorphism. If $\tilde{P}_i$ is a closed invariant tubular neighborhood of $\tilde{P}_i$ and $\pi: \hat{P}_i = \partial \tilde{P}_i \rightarrow \tilde{P}_i$ the equivariant map which is a bundle with fibre $S_x^1 = \pi^{-1}(x)$, $x \in \tilde{P}_i$, then $H_1(S_x^1; Z_m) \rightarrow H_1(\hat{P}_i; Z_m) \rightarrow H_1(\tilde{M} \setminus \tilde{P}_i; Z_m)$ is an isomorphism for each x and it is exactly the isomorphism $H_2(\tilde{M}; \tilde{M} \setminus \tilde{P}_i; Z_m) \rightarrow H_1(\tilde{M} \setminus \tilde{P}_i; Z_m)$. But since $\tilde{P}_i$ consists of fixed points, S_x^1 is a cycle of type Σl for a special simplex l , therefore its homology class mod m is the same as that of any other cycle $\Sigma l'$ with l' special (as in the proof of Theorem A). If we choose l' to be the degenerate special simplex which lies in $\tilde{P}_2$ if $i = 1$ or $\tilde{P}_1$ if $i = 2$, clearly the homology class (mod m) $[\Sigma l']$ is zero which is in contradiction with our assumption that $H_2(\tilde{M}, \tilde{M} \setminus \tilde{P}_i; Z_m) = Z_m \rightarrow H_1(\tilde{M} \setminus \tilde{P}_i; Z_m)$ is an isomorphism. Consequently ψ is an isomorphism and therefore $H_1(\tilde{M} \setminus \tilde{P}_i; Z_m) = 0$ and $H_2(\tilde{M} \setminus \tilde{P}_i; Z_m) = 0$.

To prove (2) let μ_ϵ , $\epsilon = m$ or r , be the restriction of the action of μ to Z_ϵ . Since μ_ϵ has fixed points, let $\tilde{\mu}_\epsilon$ be liftings of μ_ϵ to the actions $\tilde{\mu}_\epsilon: Z_\epsilon \rightarrow \tilde{M} \rightarrow \tilde{M}$ with $f^{-1}(M^{Z_\epsilon}) = \tilde{M}^{Z_\epsilon}$. Notice that $\tilde{M}^{Z_m}$ is invariant for $\tilde{\mu}_r$ because M^{Z_m} is invariant for μ_r . If $G_{m,r} = Z_m \rtimes_r Z_r$, then $\tilde{M}^{Z_r}$ is invariant for $\tilde{\mu}_m$. Assume $m = \infty$ and let $\tilde{P}_1$ and $\tilde{P}_2$ resp. $\tilde{R}_1$ and $\tilde{R}_2$ be the connected components of $\tilde{M}^{Z_m}$ resp. $\tilde{M}^{Z_r}$. By Smith theory we know that $\tilde{P}_i$ have trivial homology for any coefficients; we also know that $\tilde{P}_i$ is invariant to $\tilde{\mu}_r$ and the $(\tilde{R}_1 \cup \tilde{R}_2) \cap \tilde{P}_i = \tilde{P}_i^{Z_r}$. Since $H_*(\tilde{P}_i^{Z_r}; Z_n) = H_*(M; Z_n)$, $\tilde{P}_i^{Z_r}$ is nonempty so $\tilde{P}_i$ intersects at least one of the components $\tilde{R}_1$ or $\tilde{R}_2$, say $\tilde{R}_1$. Moreover if $\tilde{R}_1 \cap \tilde{P}_i \neq \tilde{P}_i$, $\tilde{R}_1 \cap \tilde{P}_i$ has to be of dimension at most $n - 2$. Since $\tilde{R}_1 \cap \tilde{P}_i \rightarrow \tilde{R}_1 \cap \tilde{P}_i$ is a principal covering with group π , one concludes that $H_n(K(\pi, 1); Z_n) = 0$. But as we have seen in the proof of (1) this is not possible, hence $\tilde{R}_1 \cap \tilde{P}_i = \tilde{P}_i$. Since $\tilde{R}_1 \supset \tilde{P}_i$, and both are closed connected submanifolds of dimension n , we have $R_1 = P_i$, which shows that $\tilde{M}^{Z_m} = \tilde{M}^{Z_n}$. Of course this is not possible because this will imply that $G_{m,r}$ acts without fixed points on the fibre S_x^1 of the boundary of a tubular neighborhood of $M^{G_{m,r}}$. If $r = \infty$ then $G_{m,r} = Z_m \times S^1$

which is a group of the type $G_{\infty, m}$ previously considered. Now assume m and r are finite. We apply Corollary A' to $\tilde{M} \setminus \tilde{R}_i$ equipped with the action μ_r ($\tilde{M} \setminus \tilde{R}$ as M and $\tilde{R}_j$ as P , $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$ and $\tilde{P}_j$ as R); we conclude that $\tilde{P}_i \cap \tilde{R}_j \neq \emptyset$, and the dimension of each connected component is $\geq n - 2$. Since $\tilde{P}_i^{\mathbb{Z}_r} = \tilde{P}_i \cap (\tilde{R}_1 \cup \tilde{R}_2)$, then either $\tilde{R}_j = \tilde{P}_i$ or $\dim(\tilde{P}_i \cap \tilde{R}_j) \leq n - 2$; hence the dimension of each component of $\tilde{P}_i \cap \tilde{R}_j$ is either n or $n - 2$. In the first case $\tilde{R}_j = \tilde{P}_i$ which implies that $G_{m, r}$ acts freely on S^1 which is possible only if $G_{m, r}$ is a cyclic group, hence $G_{m, r} = \mathbb{Z}_{mr}$. The same conclusion holds if $M^{\mathbb{Z}_m} = M^{\mathbb{Z}_n}$. If $m = r$, $M^{\mathbb{Z}_m} \cap M^{\mathbb{Z}_r}$ cannot have components of dimension $n - 2$ because if they have, then $\tilde{M}^{\mathbb{Z}_m} \cap \tilde{M}^{\mathbb{Z}_r}$ has components whose homology with $\mathbb{Z}_m$ coefficients is trivial by Smith theory and consequently, $H_n(K(\pi, 1); \mathbb{Z}_m) = 0$; clearly this is not the case because of (1).

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