IMMERSIONS OF REAL FLAG MANIFOLDS

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ABSTRACT. This paper shows that, in many cases, Lam's immersions of real flag manifolds are best possible.

1. Introduction. In [2], K. Y. Lam proved

THEOREM. *The real flag manifold*

$RF(n_1, n_2, \dots, n_s) = O(n_1 + n_2 + \dots + n_s)/O(n_1) \times O(n_2) \times \dots \times O(n_s)$
can be immersed in Euclidean space with codimension $\frac{1}{2}\sum n_i(n_i - 1)$, provided this codimension is nonzero.

The purpose of this note is to observe that this result is frequently best possible. Specifically, one has

PROPOSITION. *If $\omega = \{n_1, \dots, n_s\}$ can be partitioned as $\omega = \omega_1 \cup \omega_2 \cup \omega_3$ where $\omega_1, \omega_2 \neq \emptyset$,*

(1) $|\sum_{n \in \omega_1} n - \sum_{n \in \omega_2} n| \leq 1$, and
(2) $m \in \omega_3$ implies $m \leq \sum_{n \in \omega_1} n + \sum_{n \in \omega_2} n + 1$, then Lam's result is best possible for $RF(\omega)$.

For Grassmannians, $RF(p, q)$, this result was proved by Hiller and myself [1], and the hypothesis is simply that $q = p - 1$, p , or $p + 1$. In fact, the proof is a reduction to the Grassmannian case.

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2. The proof. According to the argument given by Lam [2, p. 313] the normal bundle of $RF(n_1, \dots, n_s)$ is realized by

$$\eta = \lambda^2 \xi_1 \oplus \lambda^2 \xi_2 \oplus \dots \oplus \lambda^2 \xi_s,$$

where ξ_i is the standard n_i -plane bundle, and λ^2 denotes the second exterior power. The goal is to prove that the top-dimensional Stiefel-Whitney class of this bundle η is nonzero.

LEMMA 1. *If ω refines ω' and if the top normal Stiefel-Whitney class of $RF(\omega')$ is nonzero, then the top normal Stiefel-Whitney class of $RF(\omega)$ is nonzero.*

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PROOF. Inductively, it suffices to consider $\omega = \{n_1, n_2, \dots, n_s\}$ and $\omega' = \{n_1 + n_2, n_3, \dots, n_s\}$ adding together two of the n_i . One then has a fibration $\pi: RF(\omega) \rightarrow RF(\omega')$ with fiber $RF(n_1, n_2)$, with π^* monic in mod 2 cohomology (since the cohomology injects into that of $RF(1, \dots, 1)$). The map π splits the $(n_1 + n_2)$ -plane bundle over $RF(\omega')$ into $\xi_1 \oplus \xi_2$ and hence

$$\pi^*(\eta_{\omega'}) = \lambda^2(\xi_1 \oplus \xi_2) \oplus \lambda^2 \xi_3 \oplus \dots \oplus \lambda^2 \xi_s = (\xi_1 \otimes \xi_2) \oplus \eta_{\omega}.$$

Thus $w_{\text{top}}(\eta_{\omega}) \cdot w_{\text{top}}(\xi_1 \otimes \xi_2) = \pi^* w_{\text{top}}(\eta_{\omega'}) \neq 0$ and so $w_{\text{top}}(\eta_{\omega}) \neq 0$. $\square$

If one applies this lemma to the Proposition, adding together the integers in ω_1 and the integers in ω_2 , one is reduced to consideration of $RF(p, q, m_1, \dots, m_t)$ where $q = p - 1, p$, or $p + 1$ and each $m_i \leq p + q + 1$.

LEMMA 2. If $n_1 \leq n_2 + \dots + n_s + 1$, $s \geq 3$, and the top normal Stiefel-Whitney class of $RF(n_2, \dots, n_s)$ is nonzero, then the top normal Stiefel-Whitney class of $RF(n_1, n_2, \dots, n_s)$ is nonzero.

PROOF. Consider the map $RF(1, \dots, 1) \xrightarrow{\pi} RF(n_1, n_2, \dots, n_s)$ so that

$$w(\pi^* \xi_1) = \prod_1^{n_1} (1 + e_i), \dots, w(\pi^* \xi_j) = \prod_1^{n_j} (1 + e_{n_1 + \dots + n_{j-1} + i}), \dots$$

writes each Stiefel-Whitney class in terms of one-dimensional classes. Here $RF(1, \dots, 1) = \text{Flag}(R^{n_1 + \dots + n_s})$, for which the mod 2 cohomology was described in [1]. One then has

$$\pi^* w_{\text{top}}(\eta) = \prod_{i=1}^s s_{(n_i-1, \dots, 1)}(e_{n_1 + \dots + n_{i-1} + 1}, \dots, e_{n_1 + \dots + n_{i-1} + n_i})$$

by applying the lemma on p. 364 of [1]. Let $m = n_2 + \dots + n_s$ and

$$a = e_1^{n_1 + m - 1} e_2^{n_1 + m - 3} \dots e_j^{n_1 + m - 2j + 1} \dots e_{n_1}^{n_1 + m - 2n_1 + 1}.$$

Then

$$a \cdot \pi^* w_{\text{top}}(\eta) = e^{n_1 + m - 1} \dots e_{n_1}^{n_1 + m - n_1} \left[\prod_{i=2}^s s_{(n_i-1, \dots, 1)}(e_{n_1 + \dots + n_{i-1} + 1}, \dots) \right],$$

which is nonzero if the second factor is nonzero in $\text{Flag}(R^m)$. $\square$

Applying this lemma to remove the integers m_i from ω_3 , one reduces the Proposition to the case $RF(p, q)$ where $q = p - 1, p$, or $p + 1$. This in turn was proved in [1], to give the Proposition.

COMMENT. There are precisely four $\omega = \{n_1, \dots, n_s\}$ having each $n_i \leq 4$ to which the Proposition does not apply. They are $\{3, 1\}$, $\{4, 1\}$, $\{4, 2\}$ and $\{4, 1, 1\}$, and each has an improved immersion. RP^3 immerses in R^4 , RP^4 in R^7 , $RF(4, 2)$ is an oriented 8-dimensional manifold with $\bar{w}_2 \bar{w}_6 = 0$ and immerses in R^{14} by [3, Theorem (4.2.1)], while $RF(4, 1, 1) \subset RP^5 \times RP^5$ immerses in $R^7 \times R^7 = R^{14}$. These improved immersions are best possible, since $\bar{w}_3$, $\bar{w}_6$ and $\bar{w}_5$ are nonzero for the three latter cases.

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