ON THE DIMENSIONS OF PROJECTIONS OF COMPACT SUBSETS OF $\mathbf{R}^m$

Y. STERNFELD

ABSTRACT. In this note we continue the study of a problem considered earlier by G. Nöbling, S. Mardešić, C. Pixley and the author.

Let W be an n -dimensional compact subset of $\mathbf{R}^m$, and let $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, m\}$. It is shown that if the projection $P_{i_1, i_2, \dots, i_k}$ satisfies a certain condition called normality on W , then there exist $i_{k+1}, i_{k+2}, \dots, i_n$ so that $\dim P_{i_1, i_2, \dots, i_k, i_{k+1}, \dots, i_n}(W) = n$. It is also shown that at least $\binom{n}{k}$ such normal projections do exist (for $1 \leq k \leq n$), and an example is constructed to show that the normality condition is not necessary.

Let $\mathbf{R}^m = R_1 \times R_2 \times \dots \times R_m$ be the m -dimensional Euclidean space, and let W be an n -dimensional compact subset of $\mathbf{R}^m$. In this note we shall be interested in the question when does a canonical projection of $\mathbf{R}^m$ on one of its n -dimensional coordinate subspaces map W to an n -dimensional set. First we mention some known facts.

Set $[m] = \{1, 2, \dots, m\}$. Given a subset $a = \{i_1, i_2, \dots, i_k\}$ of $[m]$, let $P_a: \mathbf{R}^m \rightarrow \mathbf{R}^{|a|}$ be the projection $P_a(x_1, x_2, \dots, x_m) = (x_{i_1}, x_{i_2}, \dots, x_{i_k})$ where $|a| = k$ is the cardinality of a .

The following theorem was proved by G. Nöbling [5] and S. Mardešić [4] by different methods.

THEOREM. *Let W be an n -dimensional compact subset of $\mathbf{R}^m$. Then there exists an $a \subset [m]$, $|a| = n$ such that $\dim P_a(W) = n$.*

In [7] I generalized this result as follows.

THEOREM. *Let $W \subset \mathbf{R}^m$ be an n -dimensional Cantor manifold. Let $a \subset [m]$ with $|a| = 1$ be such that $\dim P_a(W) = 1$. Then there exists a $b \subset [m] \setminus a$, $|b| = n - 1$ so that $\dim P_{a \cup b}(W) = n$.*

Recall that a compact metric space W is called an n -dimensional Cantor manifold if $\dim W = n$, and no closed subset of dimension $\leq n - 2$ separates W . This result generalizes the Nöbling-Mardešić Theorem since each compact n -dimensional metric space contains an n -dimensional Cantor manifold [1, p. 93].

In [7] I also posed the following question which naturally arises from the above Theorem. Let $W \subset \mathbf{R}^m$ be an n -dimensional Cantor manifold, and let $a \subset [m]$ with

$|a| = k \leq n \leq m$ be such that $\dim P_a(W) = k$. Does it follow that there exists $b \subset [m] \setminus a$, $|b| = n - k$ so that $\dim P_{a \cup b}(W) = n$?

A negative answer to this question was given by C. Pixley in [6]. He constructed a 3-cell $W \subset \mathbf{R}^4$ so that $\dim P_{12}(W) = \dim P_{123}(W) = \dim P_{124}(W) = 2$ and generalized this counterexample to all possible values of $2 \leq k < n < m$.

In this note we define a condition on P_a (called normality) which, if satisfied, implies a positive answer to the above question. It also provides a better understanding of the nature of Pixley's example.

DEFINITION. Let W be an n -dimensional subset of $\mathbf{R}^m$, and $a \subset [m]$ with $1 \leq |a| = k \leq n$. We say that P_a is normal on W if

$$\dim\{y \in P_a(W) : \dim(P_a^{-1}(y) \cap W) \geq n - k\} = k.$$

Note that if P_a is normal on W then $\dim P_a(W) = k$. We shall see later that the converse is false. However, for $k = n$ the converse statement holds clearly too, i.e., for $k = n$, P_a is normal on W if and only if $\dim P_a(W) = n$.

The usefulness of the concept of normality and the fact that (many) normal projections actually exist follow from the following two theorems.

THEOREM 1. *Let W be an n -dimensional compact subset of $\mathbf{R}^m$, and let $a \subset [m]$ with $1 \leq |a| = k < n$ be such that P_a is normal on W . Then there exists $b \subset [m] \setminus a$, $|b| = n - k$ so that $\dim P_{a \cup b}(W) = n$.*

THEOREM 2. *Let W be an n -dimensional compact subset of $\mathbf{R}^m$. For $1 \leq k \leq n$ set*

$$A_k = \{a \subset [m] : |a| = k, P_a \text{ is normal on } W\}.$$

Then $|A_k| \geq \binom{n}{k}$.

Note that by Theorem 1, the projection P_{12} in Pixley's example is not normal on W , although $\dim P_{12}(W) = 2$. Indeed, if P_{12} were normal on W , then by Theorem 1 one of the equalities $\dim P_{123}(W) = 3$ or $\dim P_{124}(W) = 3$ must have held. Later, we shall see another example of this kind.

PROOF OF THEOREM 1. Set

$$Y = \{y \in P_a(W) : \dim(P_a^{-1}(y) \cap W) \geq n - k\}$$

by hypothesis, Y is a k -dimensional subset of $\mathbf{R}^k$ and hence [1, p. 44] contains a k -cell E . For $b \subset [m] \setminus a$, $|b| = n - k$ set $T_b = \{y \in E : \dim P_b(P_a^{-1}(y) \cap W) = n - k\}$. Then

$$E = \bigcup_{\substack{b \subset [m] \setminus a \\ |b| = n - k}} T_b.$$

Indeed let $y \in E$, then $\dim(P_a^{-1}(y) \cap W) \geq n - k$, hence by the Nöbling-Mardešić Theorem there exists a $b \subset [m] \setminus a$, $|b| = n - k$, such that $\dim P_b(P_a^{-1}(y) \cap W) = n - k$ i.e., $y \in T_b$.

(REMARK. Theorem 1 can be proved without using the Nöbling-Mardešić Theorem. But since such a proof is longer and might hide the basic ideas, we choose the shorter one.)

It follows that there exists some $b \subset [m] \setminus a$, $|b| = n - k$, so that T_b is of second category in E . Keep this b fixed from now on. Let $\{B_l\}_{l=1}^\infty$ be a sequence of $(n - k)$ -cells whose interiors form a basis for the topology of $\mathbf{R}^{[b]}$. Set $T_{b,l} = \{y \in T_b : P_b(P_a^{-1}(y) \cap W) \supset B_l\}$. Then $T_b \subset \bigcup_{l \geq 1} T_{b,l}$. Indeed, if $y \in T_b$, then $\dim P_b(P_a^{-1}(y) \cap W) = n - k$ and hence $P_b(P_a^{-1}(y) \cap W)$ has a nonempty interior in $\mathbf{R}^{[b]}$ and it follows that $P_b(P_a^{-1}(y) \cap W)$ contains a B_l for some $l \geq 1$, i.e., $y \in T_{b,l}$.

Since T_b is of second category in E , it follows that for some $l \geq 1$ the interior of $\overline{T_{b,l}}$ in E is not empty, and thus contains a k -cell $D \subset \overline{T_{b,l}}$. Keep this l fixed.

Clearly, $T_{b,l} \times B_l \subset P_{a \cup b}(W)$. Indeed, let $(y, v) \in T_{b,l} \times B_l$, since

$$P_b(P_a^{-1}(y) \cap W) \supset B_l,$$

there exists some $w \in P_a^{-1}(y) \cap W$ such that $P_b(w) = v$ and then $P_{a \cup b}(w) = (y, v)$. By the compactness of $P_{a \cup b}(W)$ we also have

$$D \times B_l \subset \overline{T_{b,l} \times B_l} \subset \overline{T_{b,l}} \times B_l \subset P_{a \cup b}(W),$$

i.e., $P_{a \cup b}(W)$ contains the n -cell $D \times B_l$, and the theorem follows. $\square$

Before proceeding to the proof of Theorem 2, let us remark that the condition in Theorem 1 is not necessary, i.e., it is possible that $\dim P_{a \cup b}(W) = n$ for $a, b \subset [m]$, $a \cap b = \emptyset$ and $|a| + |b| = n$ even though P_a is not a normal projection on W . (But clearly $P_a(W)$ must be $|a|$ dimensional.) An intuitive explanation to such a phenomena can be as follows: suppose $W \subset \mathbf{R}^m$ is n -dimensional and compact and $\dim P_a(W) = n - 1$ for some $a \subset [m]$ with $|a| = n - 1$. Let also P_a be nonnormal on W , i.e., $\dim\{y \in P_a(W) : \dim(P_a^{-1}(y) \cap W) \geq 1\} < n - 1$, but suppose that for all $y \in P_a(W)$, $P_a^{-1}(y) \cap W$ contains some zero-dimensional set Δ_y , such that $P_i(\Delta_y) = I$ a fixed segment in R_i , for some $i \in [m] \setminus a$. It is easily seen that in this case $P_{a \cup \{i\}}(W) \supset P_a(W) \times I$, i.e. $\dim P_{a \cup \{i\}}(W) = n$.

Examples of this type do exist. We shall describe one after the proof of Theorem 2. Let us also remark that if the condition of normality in Theorem 1 were necessary, then Theorem 2 would have been trivial; indeed, let $b \subset [m]$ with $|b| = n$ be such that $\dim P_b(W) = n$ (such a b exists by the Nöbling-Mardešić Theorem), the converse of Theorem 1 then would imply that for all $a \subset b$ such that $|a| = k$, P_a is normal on W and there are $\binom{n}{k}$ such a 's. The fact that still there exist at least $\binom{n}{k}$ such a 's seem therefore interesting.

PROOF OF THEOREM 2. We use induction on k . Let W be an n -dimensional compact subset of $\mathbf{R}^m$. Consider first the case $k = 1$. Let $W' \subset W$ be an n -dimensional Cantor manifold. Set $B_1 = \{i \in [m] : P_i \text{ is not constant on } W'\}$. Then $|B_1| \geq n$ since P_{B_1} is a homeomorphism on W' . Also $B_1 \subset A_1 = \{i \in [m] : P_i \text{ is normal on } W'\}$.

Indeed, let $i \in B_1$. Then $P_i(W')$ is a nondegenerate segment $[r, s]$ say. For $r < t < s$, t separates $P_i(W')$ and hence $P_i^{-1}(t) \cap W'$ separates W' . Since W' is an n -dimensional Cantor manifold $\dim(P_i^{-1}(t) \cap W') \geq n - 1$ for $t \in (r, s)$, thus

$$\{t \in P_i(W') : \dim(P_i^{-1}(t) \cap W') \geq n - 1\} \supset (r, s)$$

i.e., P_i is normal on W .

Assume now that $|A_l| \geq \binom{n}{l}$ for all $1 \leq l \leq k$; we must prove $|A_{k+1}| \geq \binom{n}{k+1}$. Note that the case $k+1 = n$ is trivial, so we may assume $k+1 \leq n-1$ or $k \leq n-2$. In the proof to come we shall use the notation of n -dimensional degree of a subset of a separable metric space M which is defined as follows: Let X be a subset of a separable metric space M . The n -dimensional degree of X (with respect to M) denoted by $d_n(X)$ is the infimum of the numbers $\varepsilon > 0$ for which there exists a finite family F of open subsets of M , so that F covers X , the diameter of each element of F is $< \varepsilon$, and the intersection of any $n+1$ elements of F is empty. (See [3, p. 105].) If X is compact then $\dim X \leq n$ is equivalent to $d_{n+1}(X) = 0$ [3, p. 106]. (In the following, $d_n(\cdot)$ will always be taken with respect to $\mathbf{R}^m$.)

Now let a be an element of A_k . Let

$$Y = \{y \in P_a(W) : \dim(P_a^{-1}(y) \cap W) \geq n - k\}.$$

Then $\dim Y = k$. For each $i \in [m] \setminus a$, let $\{B_{i,l}\}_{l=1}^\infty$ be a sequence of closed segments in $R_i = P_i(\mathbf{R}^m)$ so that the interiors of the $B_{i,l}$'s form a basis for the topology of R_i , and let ε be positive. Set

$$T_{i,l,\varepsilon} = \{y \in Y : d_{n-k-1}(P_a^{-1}(y) \cap P_i^{-1}(t) \cap W) \geq \varepsilon \text{ for all } t \in B_{i,l}\}.$$

For a set $b \subset [m] \setminus a$, with $|b| \geq n - k$, and a finite sequence $L_b = \{l_i\}_{i \in b}$ of positive integers set

$$T_{b,L_b,\varepsilon} = \bigcap_{\substack{i \in b \\ l = l_i}} T_{i,l,\varepsilon}.$$

(Note that to each b there correspond countably many L_b 's.)

We claim that

(I) $Y = \bigcup T_{b,L_b,1/p}$ where the union is taken over all subsets b of $[m] \setminus a$, with $|b| \geq n - k$, all $L_b = (l_i)_{i \in b}$ and all natural p .

(II) $T_{b,L_b,\varepsilon}$ is a closed subset of Y .

Let us show first how the theorem follows from (I) and (II), and then prove (I) and (II).

By (I) and (II) the k -dimensional subset Y of $P_a(W)$ is a countable union of the closed sets $T_{b,L_b,1/p}$. Hence for some b , L_b , and p we must have $\dim T_{b,L_b,1/p} = k$. Fix such a b , $L_b = (l_i)_{i \in b}$ and $1/p = \varepsilon$. (Recall that $|b| \geq n - k$.) Let $i \in b$. Then for all $y \in T_{b,L_b,\varepsilon}$ and all $t \in B_{i,l_i}$, $d_{n-k-1}(P_a^{-1}(y) \cap P_i^{-1}(t) \cap W) \geq \varepsilon$ and since $P_a^{-1}(y) \cap P_i^{-1}(t) \cap W$ is compact also $\dim(P_a^{-1}(y) \cap P_i^{-1}(t) \cap W) \geq n - k - 1$ holds. But $P_a^{-1}(y) \cap P_i^{-1}(t) \cap W = P_{a \cup \{i\}}^{-1}(y, t) \cap W$. Hence, for all $z \in T_{b,L_b,\varepsilon} \times B_{i,l_i}$, $\dim(P_{a \cup \{i\}}^{-1}(z) \cap W) \geq n - k - 1$, and $T_{b,L_b,\varepsilon} \times B_{i,l_i}$ is a $(k+1)$ -dimensional subset of $\mathbf{R}^{|a|} \times R_i$ (since $T_{b,L_b,\varepsilon}$ is a k -dimensional subset of $\mathbf{R}^{|a|}$), i.e., $P_{a \cup \{i\}}$ is normal on W . It follows that A_{k+1} contains $a \cup \{i\}$ for all $i \in b$, and since a was an arbitrary element of A_k , it follows that for each $a \in A_k$ there exists a subset $b = b_a$ of $[m] \setminus a$ with $|b_a| \geq n - k$ such that for all $i \in b_a$, $a \cup \{i\} \in A_{k+1}$. This procedure contributes $|A_k| \cdot (n - k)$ elements to A_{k+1} , and since each such element is counted at most $k+1$ times we obtain

$$|A_{k+1}| \geq |A_k| (n - k) / k + 1 \geq \binom{n}{k} (n - k) / k + 1 = \binom{n}{k+1}$$

and we are done.

Now we prove (I) and (II).

PROOF OF (I). Let $y \in Y$. Then $\dim(P_a^{-1}(y) \cap W) \geq n - k$. Hence $P_a^{-1}(y) \cap W$ contains an $(n - k)$ -dimensional Cantor manifold W_y . Set $b = b_y = \{i \in [m] \setminus a : P_i \text{ is not constant on } W_y\}$. Then $|b| \geq n - k$ (since P_b is a homeomorphism on W_y). Let $i \in b$. Then $P_i(W_y)$ is a nondegenerate interval in R_i . Set $D_{i,p} = \{t \in P_i(W_y) : d_{n-k-1}(W_y \cap P_i^{-1}(t)) \geq 1/p\}$. Then $\text{int } P_i(W_y) = \bigcup_{p \geq 1} D_{i,p}$, and $D_{i,p}$ is closed in R_i . To see $\text{int } P_i(W_y) = \bigcup_{p \geq 1} D_{i,p}$, let $t \in \text{int } P_i(W_y)$. Then t separates $P_i(W_y)$, hence $P_i^{-1}(t) \cap W_y$ separates W_y , and since W_y is an $(n - k)$ -dimensional Cantor manifold $\dim(P_i^{-1}(t) \cap W_y) \geq n - k - 1$. Since this set is compact, we also have $d_{n-k-1}(P_i^{-1}(t) \cap W_y) > 0$, and it follows that for some $p \geq 1$, $d_{n-k-1}(P_i^{-1}(t) \cap W_y) \geq 1/p$, i.e., $t \in D_{i,p}$.

To see $D_{i,p}$ is closed, let $\{t_r\}_{r=1}^\infty \subset D_{i,p}$, such that $t_r \rightarrow t_0$, as $r \rightarrow \infty$, and assume $t_0 \notin D_{i,p}$, that is $d_{n-k-1}(P_i^{-1}(t_0) \cap W_y) < 1/p$. Hence, by the definition of d_{n-k-1} there exists a finite family F of $\mathbf{R}^m$ open subsets, with diameters $< 1/p$, which cover $P_i^{-1}(t_0) \cap W_y$, so that no $n - k$ elements of F intersect. (Recall that we consider d_{n-k-1} with respect to $\mathbf{R}^m$.) From the continuity of P_i on W_y it follows that $P_i^{-1}(\cdot) \cap W_y$ is uppersemicontinuous, and since F covers $P_i^{-1}(t_0) \cap W_y$, and $t_r \rightarrow t_0$, F covers also $P_i^{-1}(t_r) \cap W_y$ for sufficiently large r . Thus $d_{n-k-1}(P_i^{-1}(t_r) \cap W_y) < 1/p$ too—contradicting the fact that $t_r \in D_{i,p}$, and it follows that $D_{i,p}$ is closed.

From the fact that $\text{int } P_i(W_y)$ is the countable union of the closed sets $D_{i,p}$, it follows that for some $p_i \geq 1$ the interior of D_{i,p_i} in R_i is not empty, and in particular D_{i,p_i} contains one of the segments $B_{i,l}$, B_{i,l_i} say. Hence, for $t \in B_{i,l_i}$ we have $d_{n-k-1}(P_i^{-1}(t) \cap W_y) \geq 1/p_i$, and clearly also $d_{n-k-1}(P_a^{-1}(y) \cap P_i^{-1}(t) \cap W) \geq 1/p_i$ (since $W_y \subset P_a^{-1}(y) \cap W$). Thus $y \in T_{i,l_i,1/p_i}$. Let now $p = \max\{p_i\}_{i \in b}$, $L_b = \{l_i\}_{i \in b}$. Obviously also $y \in T_{i,l_i,1/p}$. Hence

$$y \in \bigcap_{\substack{i \in b \\ l = l_i}} T_{i,l,1/p} = T_{b,L_b,1/p}$$

and (I) follows.

PROOF OF (II). $T_{b,L_b,\epsilon}$ is an intersection of sets of the form $T_{i,l,\epsilon}$, which themselves are of the form

$$\begin{aligned} T_{i,l,\epsilon} &= \{y \in Y : d_{n-k-1}(P_a^{-1}(y) \cap P_i^{-1}(t) \cap W) \geq \epsilon \text{ for all } t \in B_{i,l}\} \\ &= \bigcap_{t \in B_{i,l}} \{y \in Y : d_{n-k-1}(P_a^{-1}(y) \cap P_i^{-1}(t) \cap W) \geq \epsilon\}. \end{aligned}$$

Hence, to prove that $T_{b,L_b,\epsilon}$ is closed it suffices to prove that $\{y \in Y : d_{n-k-1}(P_a^{-1}(y) \cap P_i^{-1}(t) \cap W) \geq \epsilon\}$ is closed. But this is completely analogous to the proof that the sets $D_{i,p}$ were closed. (There y was fixed, and we applied the continuity of P_i . Here t is fixed and one uses the continuity of P_a .) We omit the obvious details. This completes the proof of Theorem 2. $\square$

Example of a topological 3-cell $W \subset \mathbf{R}^6$ such that $P_{123}(W)$ is a 3-cell but P_{12} is not a normal projection on W .

As mentioned after the proof of Theorem 1 the example will be of the following type: we shall have $P_{12}(W) = I^2$, where I is the interval $[0, 1]$ in $\mathbf{R}$. P_{12} is not normal

on W , i.e., $\dim\{y \in I^2: \dim(P_{12}^{-1}(y) \cap W) \geq 1\} = 1$, but for each $y \in I^2$, $P_{12}^{-1}(y) \cap W$ contains a Cantor set Δ_y which is mapped onto I by P_3 , and it follows that $P_{123}(W) = I^3$.

The construction will be carried out in several steps.

Step 1. There exists a mapping $\tau: I^2 \rightarrow I^2$ with the following properties:

(1.1) There exists a Cantor set $\Delta \subset I \times \{0\}$ such that $\tau(\Delta) = I^2$.

(1.2) For every compact $K \subset I^2 \setminus \Delta$, $\tau(K)$ is contained in a finite union of straight line segments.

From (1.2) it follows that

(1.3) $\tau(I^2 \setminus \Delta)$ is contained in a countable union of segments, and hence is 1-dimensional.

The mapping τ has been constructed by C. Pixley in [6, p. 121]. We shall refer to it as Pixley's map.

Step 2. Let Δ denote the Cantor set. There exist mappings $\mu_0: \Delta \rightarrow \Delta$ and $\varphi_0: \Delta \rightarrow I$ such that

(2.1) for each $t \in \Delta$, $\varphi_0(\mu_0^{-1}(t)) = I$.

To describe μ_0 and φ_0 , let us write $\Delta = \Delta_1 \times \Delta_2$ where Δ_1 and Δ_2 are both homeomorphic to Δ . Let $x = (u, v) \in \Delta$ with $u \in \Delta_1$ and $v \in \Delta_2$. Define $\mu_0(x) = u$. Then μ_0 maps Δ onto Δ , and for each $t \in \Delta$, $\mu_0^{-1}(t) = \{t\} \times \Delta_2$. Let $\hat{\varphi}_0: \Delta_2 \rightarrow I$ be an onto mapping, and define $\varphi_0(u, v) = \hat{\varphi}_0(v)$. Hence, for $t \in \Delta$ we have

$$\varphi_0(\mu_0^{-1}(t)) = \varphi_0(\{t\} \times \Delta_2) = \hat{\varphi}_0(\Delta_2) = I.$$

Step 3. Consider the Cantor set $\Delta \subset I \times \{0\} \subset I^2$ of Step 1, and the mapping $\mu_0: \Delta \rightarrow \Delta$ of Step 2. In this step we extend μ_0 to a map $\mu_2: I^2 \rightarrow I^2$ so that

(3.1) $\mu_2/\Delta = \mu_0$, and

(3.2) for $u \in \Delta$, $\mu_2^{-1}(u) \subset I \times \{0\}$.

To do this, let us first apply Tietze's extension theorem to extend μ_0 to a map $\mu_1: I \times \{0\} \rightarrow I \times \{0\}$. Then for $(x, y) \in I^2 = I \times I$ set $\mu_2(x, y) = (\pi_1(\mu_1(x, 0)), y)$ (where $\pi_1(x, y) = x$). μ_2 is well defined since μ_1 is defined on $I \times \{0\}$. It extends μ_0 since if $t = (x, 0) \in \Delta \subset I \times \{0\}$ then

$$\mu_2(t) = \mu_2(x, 0) = (\pi_1(\mu_1(x, 0)), 0) = (\pi_1(\mu(x, 0)), 0) = (\mu(t), 0).$$

(3.2) also holds, since if $(x, y) \in I^2 \setminus I \times \{0\}$ i.e., $y > 0$ then $\mu_2(x, y) = (\pi_1(\mu_1(x, 0)), y) \in I^2 \setminus I \times \{0\}$. Thus no points of $I^2 \setminus I \times \{0\}$ is mapped by μ_2 into Δ . (Note that points of $I \times \{0\} \setminus \Delta$ may be mapped into Δ .)

Step 4. Let $\psi_0: I^2 \rightarrow I^2$ be defined by $\psi_0(v) = \tau(\mu_2(v))$ where τ is Pixley's map of Step 1 and μ_2 is the map of Step 3. We claim that the following hold.

(4.1) $\dim\{u \in I^2: \dim\psi_0^{-1}(U) \geq 1\} \leq 1$,

(4.2) for each $u \in I^2$, $\varphi_0(\psi_0^{-1}(u) \cap \Delta) = I$ (where Δ is the Cantor set contained in $I \times \{0\}$ and φ_0 is the map defined in Step 2).

We prove (4.2) first. Let $u \in I^2$. By (1.1) there exists a $v \in \Delta$ such that $\tau(v) = u$. Hence $\psi_0^{-1}(u) = \mu_2^{-1}(\tau^{-1}(u)) \supset \mu_2^{-1}(v) \supset \mu_0^{-1}(v) \subset \Delta$ by (3.1), and (2.1) $\varphi_0(\mu_0^{-1}(v)) = I$, from which (4.2) follows.

To prove (4.1) let $u \in I^2 \setminus \tau(I^2 \setminus \Delta)$. Then $\tau^{-1}(u) \subset \Delta$ and $\psi_0^{-1}(u) = \mu_2^{-1}(\tau^{-1}(u)) \subset \mu_2^{-1}(\Delta) \subset I \times \{0\}$ by (3.2). Hence the set $A = \{u \in I^2 \setminus \tau(I^2 \setminus \Delta) : \dim \psi_0^{-1}(u) \geq 1\}$ is countable (since for $u \in A$ $\psi^{-1}(u)$ contains a nondegenerate segment in $I \times \{0\}$, and an uncountable family as such segments cannot be mutually disjoint). It follows that

$$B = \{u \in I^2 : \dim \psi_0^{-1}(u) \geq 1\} \subset \tau(I^2 \setminus \Delta) \cup A,$$

where A is countable and $\tau(I^2 \setminus \Delta)$ is contained in a countable union of segments by (1.3). Hence $\dim B \leq 1$.

REMARK. Actually $\dim B = 1$. This follows, e.g., from [2, Theorem 5, p. 59].

Step 5. Let $Z \subset I^2 \times I \subset \mathbb{R}^3$ be the convex hull of $I^2 \times \{0\}$ and the point $(\frac{1}{2}, \frac{1}{2}, 1) \in \mathbb{R}^3$. Clearly Z is a 3-cell. Let us keep in mind the Cantor set Δ in $I \times \{0\} \subset I^2 \subset Z$. Let $\varphi: Z \rightarrow I$ be any extension of φ_0 defined on Δ in Step 2.

Define also a map $\psi: Z \rightarrow I^2$ by: for $z = (x, y) \in Z$ where $x \in I^2$ and $y \in I$, $\psi(z) = \psi_0(x)$. Note that for $u \in I^2$, $\psi^{-1}(u) = (\psi_0^{-1}(u) \times I) \cap Z$. We claim that

$$(5.1) \dim\{u \in I^2 : \dim \psi^{-1}(u) \geq 1\} \leq 1, \text{ and}$$

$$(5.2) \text{ for all } u \in I^2, \varphi(\psi^{-1}(u)) = I.$$

To prove (5.1) note that for $u \in I^2 \setminus \tau(I^2 \setminus \Delta)$

$$\psi^{-1}(u) = (\psi_0^{-1}(u) \times I) \cap Z \subset (I \times \{0\} \times I) \cap Z = I \times \{0\} \times \{0\}.$$

Thus $\{u \in I^2 : \dim \psi^{-1}(u) \geq 1\} \subset \tau(I^2 \setminus \Delta) \cup A$ where A is the set defined in the proof of (4.1), and (5.1) follows as in (4.1). Actually

$$\{u : \dim \psi^{-1}(u) \geq 1\} = B = \{u : \psi_0^{-1}(u) \geq 1\}.$$

(5.2) follows at once from (4.2) since $\psi^{-1}(u) \supset (\psi_0^{-1}(u) \times \{0\}) \cap \Delta$ hence $\varphi(\psi^{-1}(u)) \supset \varphi_0(\psi_0^{-1}(u) \cap \Delta) = I$.

Step 6. We now embed Z in $\mathbb{R}^6 = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^3$ by $h(z) = (\psi(z), \varphi(z), z)$, $z \in Z \subset \mathbb{R}^3$, $\psi(z) \in I^2 \subset \mathbb{R}^2$ and $\varphi(z) \in I \subset \mathbb{R}$. Clearly, h is an embedding. Set $W = h(Z)$. So W is a 3-cell in $\mathbb{R}^6$. We claim that

$$(6.1) P_{123}(W) = I^2 \times I \text{ is a 3-cell in } \mathbb{R}^3, \text{ and}$$

$$(6.2) P_{12} \text{ is not a normal projection on } W.$$

To prove (6.1) note first that $P_{12}(W) = \psi(Z) = I^2$ and $P_3(W) = \varphi(Z) = I$, so $P_{123}(W) \subset I^2 \times I$. Also, $P_{123}(W) = \{(\psi(z), \varphi(z)) : z \in Z\}$. Let $(u, t) \in I^2 \times I$. By (5.2) $\varphi(\psi^{-1}(u)) = I$, so there exists some $z \in \psi^{-1}(u)$ such that $\varphi(z) = t$, i.e., $(\psi(z), \varphi(z)) = (u, t)$, and it follows that $P_{123}(W) \supset I^2 \times I$.

To show that P_{12} is not normal, we must show that

$$\dim\{u \in P_{12}(W) = \psi(Z) = I^2 : \dim(P_{12}^{-1}(u) \cap W) \geq 1\} \leq 1.$$

But $P_{12}^{-1}(u) \cap W = h(\psi^{-1}(u))$, and h is an embedding hence

$$\{u \in P_{12}(W) : \dim(P_{12}^{-1}(u) \cap W) \geq 1\} = \{u \in I^2 : \dim \psi^{-1}(u) \geq 1\}$$

which has dimension ≤ 1 by (5.1). This completes the example.

Finally, let us remark, that if we omit the coordinate φ from h , i.e., consider the embedding $h': Z \rightarrow \mathbb{R}^5 = \mathbb{R}^2 \times \mathbb{R}^3$ defined by $h'(z) = (\psi(z), z)$, and set $W' = h'(Z)$,

then W' is a 3-cell in $\mathbf{R}^5$, and $\dim P_{12}(W') = 2 = \dim P_{123}(W') = \dim P_{124}(W') = \dim P_{125}(W')$ as in Pixley's example. Obviously, this construction depends heavily on Pixley's mapping τ (as Pixley's example does). Note also that if one is interested in h' and W' only, then the maps μ_1 and φ_i can be omitted, and one can take ψ to be defined as $\psi(z) = \tau(x)$, where $z = (x, y) \in Z$, $x \in I^2$, $y \in I$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, HAIFA, ISRAEL