THE RULED RESIDUE THEOREM FOR SIMPLE TRANSCENDENTAL EXTENSIONS OF VALUED FIELDS

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ABSTRACT. A proof is given for the Ruled Residue Conjecture, which asserts that if v is a valuation of a simple transcendental field extension $K_0(x)$ and v_0 is the restriction of v to K_0 , then the residue field of v is either ruled or algebraic over the residue field of v_0 .

1. Introduction. Let $K_0 \subset K = K_0(x)$ be fields with x transcendental (abbreviated tr.) over K_0 , let v_0 be a valuation of K_0 and v be an extension of v_0 to K ; and let $V_0 \subset V$, $k_0 \subset k$, and $G_0 \subset G$ be the respective valuation rings, residue fields, and value groups.

We prove here the Ruled Residue Conjecture, which asserts that k is either algebraic over k_0 or ruled over k_0 . ("Ruled" means that there should be a field k_1 with $k_0 \subset k_1 \subset k$ and k simple tr. over k_1 ; in the present setting such a k_1 is necessarily finite algebraic over k_0 . Cf. [3].)

This conjecture was suggested by Nagata [2], who proved the case of discrete, finite rank v_0 . The char. 0 case of the conjecture (i.e. char. $k_0 = 0$) was settled by the present author in [3], and Heinzer [1] subsequently improved this result to the case that k_0 is perfect. The proof of the full conjecture given below is completely elementary and direct, in contrast to the proofs of these special cases.

2. Additional notation. The notation $()^*$ will indicate image under the canonical homomorphism $V \rightarrow k$; and for $a \in K$ we shall write $a \rightarrow a^*$ to mean $a \in V$ and a^* is the image of a under the canonical homomorphism $V \rightarrow k$. Note that if $\xi \in V$ and $\xi \rightarrow \xi^*$ tr. over k_0 , then $v(\xi) = 0$ and $1/\xi \rightarrow 1/\xi^*$.

For any $f \neq 0$ in $K_0[x]$, define $\deg f$ to be n if $f = a_0 + a_1x + \cdots + a_nx^n$ with $a_i \in K_0$ and $a_n \neq 0$. Any $\xi \neq 0$ in $K_0(x)$ can be written $\xi = f/g$, where f, g are nonzero, relatively prime elements of $K_0[x]$; and such f, g are unique up to nonzero multiplies from K_0 . We shall define $\deg \xi$ to be $\max\{\deg f, \deg g\}$. Note that if $\xi \in V$ and $\xi \rightarrow \xi^*$ tr. over k_0 , then $\xi \notin K_0$ and hence $\deg \xi > 0$.

Finally, if k/k_0 is not algebraic, then there exists $\xi \in V$ such that $\xi \rightarrow \xi^*$ tr. over k_0 . In this case we shall let $S = \{\xi \in V \mid \xi \rightarrow \xi^* \text{ tr. over } k_0\}$ and $\min S = \{\xi \in S \mid \deg \xi \leq \deg \zeta \text{ for all } \zeta \in S\}$.

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3. Proof of the theorem.

3.1. LEMMA. *If there exists $\xi \in V$ of $\deg n$ such that $\xi \rightarrow \xi^*$ tr. over k_0 , then there exists $\xi_1 \in V$ such that $\xi_1 \rightarrow \xi_1^*$ tr. over k_0 and $k_0(\xi_1^*) = k_0(\xi^*)$, and such that ξ_1 has the additional property that $\xi_1 = f_1/g_1$, where f_1, g_1 are relatively prime elements of $K_0[x]$ with $\deg f_1 = n$, $\deg g_1 < n$.*

PROOF. Write $\xi = f/g$, where f, g are relatively prime elements of $K_0[x]$. If $\deg g > \deg f$, then $\xi_1 = 1/\xi$ works.

Thus, we may assume $\deg g = \deg f = n$; and we may then write $f = ag + r$, where $0 \neq a \in K_0$, $0 \neq r \in K_0[x]$, and $\deg r < n$.

If $v(a) \geq 0$, then $r/g = (f/g) - a \in V$ and $(r/g)^* = (f/g)^* - a^*$ is tr. over k_0 . Thus, $g/r = \xi_1$ works.

If $v(a) < 0$, then $r/af = (1/a) - (g/f) \in V$ and $(r/af)^* = (1/a)^* - (g/f)^*$ is tr. over k_0 . Thus, $af/r = \xi_1$ works.

3.2. LEMMA. *Suppose $\xi = f/g$, where $f, 0 \neq g \in K_0[x]$ and $n = \deg f > \deg g$. Then*

$$K_0[x] \subset K_0[\xi] + K_0[\xi]x + \cdots + K_0[\xi]x^{n-1};$$

or, equivalently, any $h \in K_0[x]$ may be written

$$h = a_m \xi^m + a_{m-1} \xi^{m-1} + \cdots + a_0,$$

where $a_i \in K_0[x]$ and either $\deg a_i < n$ or $a_i = 0$.

PROOF. Let M denote the $K_0[\xi]$ -module, $K_0[\xi] + K_0[\xi]x + \cdots + K_0[\xi]x^{n-1}$. It suffices to see $x^n \in M$; for then it follows by induction on s that $x^{n+s} \in M$ for all $s \geq 0$, and hence that $K_0[x] \subset M$.

Since $\deg f = n$, $x^n = af + r$, where $0 \neq a \in K_0$, $r \in K_0[x]$, and either $\deg r < \deg f$ or $r = 0$. Then $x^n = (ag)(f/g) + r = (ag)\xi + r$, where $\deg(ag) = \deg g < n$ and either $\deg r < n$ or $r = 0$.

3.3. RULED RESIDUE THEOREM. *Suppose k is not algebraic over k_0 , and let $\xi \in \min S$. Then $k = k'_0(\xi^*)$, where k'_0 is the algebraic closure of k_0 in k .*

PROOF. The inclusion $\supset$ is immediate, so it remains to show $\subset$. By 3.1 we may assume $\xi = f/g$, where $\deg f = n > \deg g$. For any $\zeta \in V$, we may write, by 3.2,

$$\zeta = (a_m \xi^m + a_{m-1} \xi^{m-1} + \cdots + a_0) / (b_m \xi^m + b_{m-1} \xi^{m-1} + \cdots + b_0),$$

where a_i, b_i are elements of $K_0[x]$ which are either of $\deg < n$ or are 0. Let d be an element of least value from among $\{a_i, b_i \mid i = 0, \dots, m\}$, and let $\alpha_i = a_i/d, \beta_i = b_i/d$. Then

$$\zeta = (\alpha_m \xi^m + \alpha_{m-1} \xi^{m-1} + \cdots + \alpha_0) / (\beta_m \xi^m + \beta_{m-1} \xi^{m-1} + \cdots + \beta_0),$$

where now the coefficients α_i, β_i are elements of $K_0(x)$ which are either of $\deg < n$ or are 0. Moreover, $v(\alpha_i), v(\beta_i), i = 0, \dots, m$, are all ≥ 0 . We may therefore consider the equality

$$(\beta_m^* \xi^{*m} + \cdots + \beta_0^*) \zeta^* = \alpha_m^* \xi^{*m} + \cdots + \alpha_0^*.$$

Since $\xi \in \min S$ and the α_i, β_i are either 0 or of $\deg < \deg \xi$, it follows that the α_i^*, β_i^* are all algebraic over k_0 . But ξ^* is tr. over k_0 and we know some α_i^* or β_i^* is 1, so some β_i^* must be $\neq 0$. Therefore $\beta_m^* \xi^{*m} + \cdots + \beta_0^* \neq 0$, and hence

$$\zeta^* = (\alpha_m^* \xi^{*m} + \cdots + \alpha_0^*) / (\beta_m^* \xi^{*m} + \cdots + \beta_0^*) \in k'_0(\xi^*).$$

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