RATIONAL CURVE WITH ONE CUSP

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ABSTRACT. In this paper we shall give new examples of plane curves C satisfying $C - \{P\} \cong \mathbf{A}^1$ for some point $P \in C$ by using automorphisms of open surfaces.

1. In this paper we assume the ground field is an algebraically closed field of characteristic zero. Let C be an irreducible algebraic curve in $\mathbf{P}^2$. Then let us call C a curve of type I if $C - \{P\} \cong \mathbf{A}^1$ for some point $P \in C$, and a curve of type II if $C \setminus L \cong \mathbf{A}^1$ for some line L . Of course a curve of type II is of type I. Curves of type II have been studied by Abhyankar and Moh [1] and others. On the other hand, if the logarithmic Kodaira dimension of $\mathbf{P}^2 - C$ is $-\infty$ or the automorphism group of $\mathbf{P}^2 - C$ is infinite dimensional, then C is of type I [3, 6]. So the study of type I seems to be important, but only a few examples are known that are of type I and not of type II. Moreover the definitions of those examples are not so plain [4, 5]. Here we shall give new examples by using automorphisms of open surfaces.

2. First we recall the properties of type I. Let $(e_1, \dots, e_t)$ be the sequence of the multiplicities of the infinitely near singular points of P in case C is of type I with degree $d \geq 3$. Then put

$$R = d^2 - \sum_{i=1}^t e_i^2 - e_t + 1.$$

Since a curve of type II is an image of a line by some automorphism of $\mathbf{A}^2$ [1], we see that $R \geq 2$ for this curve by the same argument as below. Denoting by G the automorphism group of $\mathbf{P}^2 - C$, we have $G \supset \mathbf{A}^m$ for all $m \geq 1$ if $R > 0$, and G is a finite group if $R < 0$ [6]. One of the conditions for type I curves to be type II is the following:

LEMMA 1. *Suppose C is type I. Then it is type II if and only if $d = e_1 + e_2$. (If $t = 1$, we put $e_2 = 1$.)*

PROOF. If C is type II, then it is a birational transform of a line [1]. Hence we have $d < e_1 + e_2 + e_3$, which is the inequality of M. Noether [2]. Moreover since C is type II, we have $d = I_P(C, L)$, which is the intersection number of C and L at P . Hence we conclude $d = e_1 + e_2$.

Conversely, assume $d = e_1 + e_2$. Note that we have, in general, $d \geq I_P(C, L) \geq e_1 + e_2$, where L is the tangent at P . Thus we have $d = I_P(C, L)$, which implies $C \cap L = \{P\}$, i.e., C is type II.

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Now we give examples. Let (X, Y, Z) be homogeneous coordinates on $\mathbf{P}^2$, and let L be the line defined by $Z = 0$. Let f be a nonlinear automorphism of $\mathbf{P}^2 - L$ and P the point of the indeterminacy of the birational transformation $\tilde{f}$ defined by f . Let H be a line not passing through P , and put $\Delta = \tilde{f}(H)$ and $n = \deg \Delta$. Then we have $n \geq 2$. Of course $\Delta \setminus L \cong \mathbf{A}^1$ and $\text{Aut}(\mathbf{P}^2 - \Delta)$ is an infinite-dimensional group.

LEMMA 2. *Any automorphism φ of $\mathbf{P}^2 - \Delta$ induces an automorphism of L fixing the point P if $n \geq 3$.*

PROOF. We consider the elimination of the point of the indeterminacy of the birational transformation $\bar{\varphi}$ defined by φ . Let σ_i , $i = 1, \dots, r$, be the blow-ups such that the total transform $\sigma^{-1}(\Delta)$, where $\sigma = \sigma_1 \cdots \sigma_r$, has normal crossings. Then the centers of them are unique if the number r is minimal. By Lemma 1 we see that $(\sigma_1 \sigma_2)^{-1}[L]$ does not meet $(\sigma_1 \sigma_2)^{-1}[\Delta]$, where $[\]$ denotes the proper transform. Since $r \geq 3$, we infer that $\bar{\varphi}[L]$ is also a line, hence it coincides with L .

Let Γ be a line or an irreducible conic meeting Δ at only one point different from P . Let g be a nonlinear automorphism of $\mathbf{P}^2 - \Delta$, and let $\bar{g}$ be the birational transformation defined by g . In case $n = 2$ we assume $\bar{g}$ induces an automorphism of L . Then put $C = \bar{g}(\Gamma)$. By the lemmas we see that $C - \{P\} \cong \mathbf{A}^1$ and C is not type II. Let σ be the composition of the blow-ups such that the number of them is the smallest one to obtain a morphism $\bar{g}\sigma$. Since the morphism $\bar{g}\sigma$ first contracts the proper transform $\sigma^{-1}[\Delta]$, we have $e_i = n$ [resp. $2n$] and $d^2 - \sum_{i=1}^r e_i^2 = 1$ [resp. 4] if $\deg \Gamma = 1$ [resp. 2]. Hence $R = 2 - n$ [resp. $5 - 2n$] if $\deg \Gamma = 1$ [resp. 2]. Moreover we have $I_p(C, L) = d - 1$ [resp. $d - 2$] if $\deg \Gamma = 1$ [resp. 2].

3. Here we give the examples more concretely. In case Γ is a line, let Δ be the curve defined by $YZ^{n-1} = X^n$. Then Γ is defined by $Y = 0$. Now, Wakabayashi shows the following: let $\bar{g}(X, Y, Z) = (g_1, g_2, g_3)$. Then

$$g_1 = aX(YZ^{n-1} - X^n)^{mn} + \sum_{i=1}^m b_i Z^{ni+1}(YZ^{n-1} - X^n)^{mn-i},$$

$$g_2 = Z^{-(n-1)} \left[a^n (YZ^{n-1} - X^n)^{mn+1} + \left\{ aX(YZ^{n-1} - X^n)^m + \sum_{i=1}^m b_i Z^{ni+1}(YZ^{n-1} - X^n)^{m-i} \right\}^n \right]$$

and

$$g_3 = Z(YZ^{n-1} - X^n)^{mn},$$

where $a \neq 0$, $m \geq 1$ and b_i are arbitrary for $i = 1, \dots, m$. Thus the curve defined by $g_2 = 0$, where $m \geq 1$ and $n \geq 2$, is the required one.

On the contrary, in case Γ is a conic, let Δ and Γ be the curves defined by $YZ - X^2 = 0$ and $YZ - X^2 + aY^2 = 0$, where $a \neq 0$, respectively.

By these examples we know that every integer ≤ 1 is taken as the value R . Finally we raise the following:

Problem. If C is type I with $R \geq 2$, then is it true that C is type II?

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