ROTATIONS AND LINKAGE OF 2-FOLD PFISTER FORMS

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ABSTRACT. We show that a pair of 2-fold Pfister forms admit rotations with the same irreducible, separable characteristic polynomial if and only if they are linked.

Baeza [1] has described the quadratic forms over a field F which admit a rotation with a given irreducible, separable characteristic polynomial. Such forms arise as the Scharlau transfer of some binary quadratic form. It is natural to ask, as Baeza does, for the relation between two quadratic forms arising this way. Here we give the answer in the case $\text{char } F \neq 2$ and the forms are 2-fold Pfister forms. Namely, a pair of 2-fold Pfister forms admit rotations with the same irreducible, separable characteristic polynomial if and only if they are linked. A partial result for other four-dimensional forms is also given.

We follow the notation of [6] for quadratic forms. The underlying field F has characteristic different from 2 and all forms are nonsingular. The construction of the Scharlau transfer and its elementary properties are in [6]. We will use these results without further mention.

A 2-fold Pfister form is the norm form of a quaternionic algebra; thus it can be denoted as

$$\langle 1, a \rangle \otimes \langle 1, b \rangle \equiv \langle \langle a, b \rangle \rangle \equiv x_1^2 + ax_2^2 + bx_3^2 + abx_4^2.$$

The *pure part* of a Pfister form $q = \langle \langle a, b \rangle \rangle$ is the form $\langle a, b, ab \rangle$ and is denoted by q' . A 2-fold Pfister form q *splits* over an extension L of F if q is hyperbolic over L , that is, $q \otimes L \sim \langle 1, -1, 1, -1 \rangle$. A pair of 2-fold Pfister forms q_1 and q_2 are *linked* if there exist $a, b, c \in \dot{F}$ such that $q_1 \simeq \langle \langle a, b \rangle \rangle$ and $q_2 \simeq \langle \langle a, c \rangle \rangle$. See [5] for a discussion of the properties (and significance) of linked Pfister forms.

Let $O(q)$ denote the orthogonal group of the quadratic form q and let $O_{\text{ir}}(q)$ denote those elements of $O(q)$ with irreducible, separable characteristic polynomials. Note that if the dimension of q is a 2-power, separability is implied by irreducibility. If $A \in O_{\text{ir}}(q)$, let $F(A)$ denote the splitting field of the characteristic polynomial of A . In the case we are interested in, namely q a 2-fold Pfister form, Edwards [2] has shown the Galois group of $F(A)/F$ is the Klein 4-group, for all $A \in O_{\text{ir}}(q)$.

We review Baeza's construction. Let q be an even-dimensional form and $A \in O_{\text{ir}}(q)$. Let $p_A(x)$ be the characteristic polynomial of A . Choose a root a of $p_A(x)$ and set $L = F(a + a^{-1})$ and $\varphi = \langle 1, -(a - a^{-1})^2 \rangle$, a form over L . Then [1, 3.7] the forms q' such that there exists an $A' \in O_{\text{ir}}(q')$ having characteristic polynomial

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equal to $p_A(x)$ are precisely the forms $s_*(\varphi)$, where $s: L \rightarrow F$ runs through all nontrivial linear functionals and s_* is the Scharlau transfer of s . If we restrict our attention to forms q' which represent 1, we need only consider those linear functionals with $s(1) = 1$.

LEMMA 1. *Let $p(x) \in F[x]$ be an irreducible polynomial arising as the characteristic polynomial of a rotation of a 2-fold Pfister form. Let K be the splitting field of $p(x)$. Then there exists a quadratic subextension E/F of K/F such that for any 2-fold Pfister form q admitting a rotation with characteristic polynomial equal to $p(x)$, q splits over E .*

PROOF. From [2] we know the roots of $p(x)$ are of the form $a, 1/a, b, 1/b$ and that $\text{Gal}(K/F)$ is the Klein 4-group. Note that one of $a - a^{-1} + b - b^{-1}$ or $a - a^{-1} - b + b^{-1}$ is nonzero, since otherwise $a - a^{-1} = 0$. We will assume that $e = a - a^{-1} + b - b^{-1}$ is not zero (the argument is the same in the other case). Then $e \notin F$ and $e^2 \in F$ by Galois theory. Set $E = F(e)$.

Now set $L = F(a + a^{-1})$ and $\varphi = \langle 1, -(a - a^{-1})^2 \rangle$. Let q be a 2-fold Pfister form admitting a rotation with characteristic polynomial equal to $p(x)$. Then $q \simeq t_*\varphi$, for some nontrivial linear functional $t: L \rightarrow F$ with $t(1) = 1$. We will show $t_*\varphi$ splits over E .

Note that $(b - b^{-1})/(a - a^{-1}) \in L$, since it is fixed by the automorphism sending $a \mapsto a^{-1}, b \mapsto b^{-1}$. Hence $e = c(a - a^{-1})$, for some $c \in L$. Then

$$\varphi \simeq \langle 1, -e^2 \rangle \quad \text{and} \quad t_*\varphi \simeq \langle 1, -e^2 \rangle \otimes t_*\langle 1 \rangle.$$

Thus over $E = F(e)$, $t_*\varphi \simeq \langle 1, -1 \rangle \otimes t_*\langle 1 \rangle$ is split.

LEMMA 2. *Let q_1 and q_2 be 2-fold Pfister forms. Suppose $A_i \in O_{\text{ir}}(q_i)$ ($i = 1, 2$) have the same characteristic polynomial. Then q_1 and q_2 are linked.*

PROOF. By Lemma 1, there is a quadratic extension E of F such that q_1 and q_2 split over E . Then q_1 and q_2 are linked [3, 4.7].

LEMMA 3. *Let q_1 and q_2 be quadratic forms and let $A_i \in O_{\text{ir}}(q_i)$, $i = 1, 2$. Suppose the eigenvalues of A_1 and A_2 are $a_1^{\pm 1}, \dots, a_r^{\pm 1}$ and $b_1^{\pm 1}, \dots, b_s^{\pm 1}$ respectively. Then*

- (1) $A_1 \otimes A_2 \in O_{\text{ir}}(q_1 \otimes q_2)$,
- (2) the eigenvalues of $A_1 \otimes A_2$ are $\{a_i^{\pm 1}b_j^{\pm 1} \mid 1 \leq i \leq r, 1 \leq j \leq s\}$.

PROOF. Straightforward.

THEOREM. *Let q_1 and q_2 be 2-fold Pfister forms. Then q_1 and q_2 are linked if and only if there exist $A_i \in O_{\text{ir}}(q_i)$ ($i = 1, 2$) with the same characteristic polynomial.*

PROOF. We need only prove necessity. We may write

$$q_1 = \langle \langle g, e_2 \rangle \rangle \quad \text{and} \quad q_2 = \langle \langle h, e_2 \rangle \rangle,$$

for some $g, h, e_2 \in \dot{F}$. We can find $j, k \in F$ such that $h - g = j^2 - k^2$. Set $e_1 = g + j^2 = h + k^2$. Then

$$q_1 \simeq \langle \langle e_1 - j^2, e_2 \rangle \rangle, \quad q_2 \simeq \langle \langle e_1 - k^2, e_2 \rangle \rangle.$$

Choose any $B_1 \in O_{\text{ir}}(\langle 1, e_2 \rangle)$ and $B_2 \in O_{\text{ir}}(\langle 1, e_1 e_2 \rangle)$. The eigenvalues of B_1 are $u \pm v\sqrt{-e_2}$ and for B_2 are $x \pm y\sqrt{-e_1 e_2}$ ($u, v, x, y \in F$). Thus by Lemma 3, $B_1 \otimes B_2 \in O_{\text{ir}}(\langle \langle e_1, e_2 \rangle \rangle)$ and $F(B_1 \otimes B_2) = F(\sqrt{e_1}, \sqrt{-e_2})$. Also the eigenvalues of $B_1 \otimes B_2$ are a, a^{-1}, b, b^{-1} with

$$a = (u + v\sqrt{-e_2})(x + y\sqrt{-e_1 e_2}) \quad \text{and} \quad b = (u + v\sqrt{-e_2})(x - y\sqrt{-e_1 e_2}).$$

Set $L = F(a + a^{-1})$ and $\varphi = \langle 1, -(a - a^{-1})^2 \rangle$; note $L = F(\sqrt{e_1})$. Consider the following two linear functionals

$$s: L \rightarrow F, \quad t: L \rightarrow F,$$

defined by

$$1 \mapsto 1, \quad \sqrt{e_1} \mapsto j, \quad 1 \mapsto 1, \quad \sqrt{e_1} \mapsto k.$$

A simple computation shows $s_* \langle 1 \rangle \simeq \langle 1, e_1 - j^2 \rangle$ and $t_* \langle 1 \rangle \simeq \langle 1, e_1 - k^2 \rangle$. Another direct computation yields

$$-(a - a^{-1} + b - b^{-1})^2 \equiv e_2 \pmod{\dot{F}^2}.$$

As in the proof of Lemma 1, we have that $(a - a^{-1} + b - b^{-1}) = c(a - a^{-1})$, for some $c \in L$ and hence that $\varphi \simeq \langle 1, e_2 \rangle$.

We thus obtain

$$s_* \varphi \simeq \langle 1, e_2 \rangle \otimes s_* \langle 1 \rangle \simeq \langle \langle e_1 - j^2, e_2 \rangle \rangle \simeq q_1,$$

$$t_* \varphi \simeq \langle 1, e_2 \rangle \otimes t_* \langle 1 \rangle \simeq \langle \langle e_1 - k^2, e_2 \rangle \rangle \simeq q_2.$$

Hence there are $C_i \in O_{\text{ir}}(q_i)$, $i = 1, 2$, with characteristic polynomial equal to that of $B_1 \otimes B_2$, by [1, 3.7].

Combining Lemma 1, the Theorem and [2] yields

COROLLARY 1. *Let q be a 2-fold Pfister form. Then an extension K of F arises as $F(A)/F$ for some $A \in O_{\text{ir}}(q)$ if and only if K is a noncyclic Galois extension of degree 4 containing a quadratic subextension E/F of K/F such that q splits over E .*

For a quadratic form q , $D(q)$ denotes the set of elements in $\dot{F}$ represented by q . If q is a 2-fold Pfister form then $D(q)$ is a subgroup of $\dot{F}$ [6, p. 278].

COROLLARY 2. *Let q_1 and q_2 be 2-fold Pfister forms. Suppose $|\dot{F}/D(q_i)| > 4$ ($i = 1, 2$). Then $D(q'_1) = D(q'_2)$ if and only if*

$$\{F(A)|A \in O_{\text{ir}}(q_1)\} = \{F(A)|A \in O_{\text{ir}}(q_2)\}.$$

PROOF. ($\rightarrow$) Let $A \in O_{\text{ir}}(q_1)$. Then $F(A) = F(\sqrt{-a}, \sqrt{-b})$ where $a \in D(q'_1)$, by Corollary 1 and the proof of Lemma 1. Since we then have $a \in D(q'_2)$, q_2 splits over the subextension $F(\sqrt{-a})/F$ of $F(A)/F$. By Corollary 1, $F(A) = F(B)$ for some $B \in O_{\text{ir}}(q_2)$.

($\leftarrow$) Suppose otherwise. We may assume there is an $a \in D(q'_1) - D(q'_2)$. Let $b \in F$ be any element not congruent to -1 or a modulo $\dot{F}^2$. We make the following

Claim. b or $-ab \in D(q'_2)$.

Now $F(\sqrt{-a}, \sqrt{-b})$ is a noncyclic Galois extension of degree 4 containing a quadratic extension, $F(\sqrt{-a})/F$, over which q_1 splits. Hence $F(\sqrt{-a}, \sqrt{-b}) \in \{F(A) \mid A \in O_{\text{ir}}(q_1)\}$. Thus $F(\sqrt{-a}, \sqrt{-b}) \in \{F(A) \mid A \in O_{\text{ir}}(q_2)\}$. Using Corollary 1 again, we see q_2 must split over $F(\sqrt{-a})$, $F(\sqrt{-b})$ or $F(\sqrt{ab})$. Hence one of a , b or $-ab$ lies in $D(q'_2)$ [6, p. 200]. By assumption, $a \notin D(q'_2)$ and so the Claim is proven.

The Claim shows that

$$\dot{F} = -\dot{F}^2 \cup a\dot{F}^2 \cup D(q'_2) \cup -aD(q'_2).$$

Hence $\dot{F} = \{\pm 1, \pm a\}D(q_2)$ and $|\dot{F}/D(q_2)| \leq 4$, a contradiction. This shows $D(q'_1) \subset D(q'_2)$ and the same argument yields the reverse inclusion.

REMARKS. (1) The restriction $|\dot{F}/D(q_i)| > 4$ in Corollary 2 is necessary. Namely, let $F = \mathbf{R}((t))$, $q_1 = \langle \langle 1, 1 \rangle \rangle$ and $q_2 = \langle \langle 1, t \rangle \rangle$. Then $D(q'_1) \neq D(q'_2)$ but

$$\{F(A) \mid A \in O_{\text{ir}}(q_1)\} = \{F(\sqrt{-1}, \sqrt{t})\} = \{F(A) \mid A \in O_{\text{ir}}(q_2)\},$$

since $F(\sqrt{-1}, \sqrt{t})$ is the only biquadratic extension of F , and both q_1 and q_2 split over $F(\sqrt{-1})$. Note that $|\dot{F}/D(q_1)| = 4$.

(2) If F is a local or global field, or a field for which I^2F , the square of the fundamental ideal of the Witt ring, is torsion-free, then $D(q'_1) = D(q'_2)$ implies $q_1 \simeq q_2$ [4, 2.2, 2.3]. In this case, Corollary 2 says the isometry class of a 2-fold Pfister form q , with $|\dot{F}/D(q)| > 4$, can be determined by examining the sets $\{F(A) \mid A \in O_{\text{ir}}(p)\}$. It is not known if there are fields F with nonisometric 2-fold Pfister forms q_1 and q_2 such that $D(q'_1) = D(q'_2)$.

We conclude with a partial result for four-dimensional forms which are not Pfister forms. First, however, we remark that if $A \in O_{\text{ir}}(q)$ then q splits over $F(A)$. Namely, a simple argument (cf. [7, p. 212]) shows that $q(v) = 0$ for each eigenvector of A with eigenvalue not equal to ± 1 . And by assumption, A has $n = \dim q$ distinct eigenvalues in $F(A)$, not in F .

COROLLARY 3. *Let q_1 and q_2 be four-dimensional forms which represent 1 and have a nonsquare discriminant d . Suppose there exist $A_i \in O_{\text{ir}}(q_i)$, $i = 1, 2$, with the same characteristic polynomial. Then over $F(\sqrt{d})$, q_1 and q_2 are linked 2-fold Pfister forms.*

PROOF. Let $p(x)$ be the characteristic polynomial of A_1 (and A_2). Let $F(A)$ be the splitting field of $p(x)$. Clearly q_1 and q_2 are 2-fold Pfister forms over $F(\sqrt{d})$ as there they have square discriminant. Also, q_1 has a square discriminant over $F(A)$ since it is hyperbolic there. Thus $F(\sqrt{d}) \subset F(A)$. And from the classification theorem of Edwards [2] we know the Galois group $G = \text{Gal}(F(A)/F)$ is either the cyclic group $\mathbf{Z}_4$ or the dihedral group D_4 .

Case 1. $G = \mathbf{Z}_4$: Here $[F(A) : F] = 4$. Since q_1 and q_2 are hyperbolic over $F(A)$, both $q_1 \otimes F(\sqrt{d})$ and $q_2 \otimes F(\sqrt{d})$ split over the quadratic extension $F(A)/F(\sqrt{d})$. Hence they are linked Pfister forms [3, 4.7].

Case 2. $G = D_4$: By Edwards' theorem, $p(x)$ is irreducible over $F(\sqrt{d})$. Thus $A_i \otimes F(\sqrt{d}) \in O_{\text{ir}}(q_i \otimes F(\sqrt{d}))$, $i = 1, 2$. In particular, $q_1 \otimes F(\sqrt{d})$ and $q_2 \otimes F(\sqrt{d})$ are 2-fold Pfister forms admitting rotations with the same irreducible characteristic polynomial. Thus $q_1 \otimes F(\sqrt{d})$ and $q_2 \otimes F(\sqrt{d})$ are linked, by the Theorem.

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