GEOMETRIC AUTOMORPHISMS OF A FREE GROUP OF RANK AT LEAST THREE ARE RARE

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ABSTRACT. The characteristic polynomials of geometric automorphisms of a free group of finite rank at least three form a nowhere dense set in the Zariski topology.

1. Introduction. Call an automorphism ϕ of a free group F of finite rank *geometric* if there exists a two dimensional manifold M , compact and bounded (but not necessarily orientable) and a homeomorphism h of the pair (M, x) , $x \in M$, such that the map on $\pi_1(M, x)$ induced by h is ϕ . Stallings [St] gave the first example of a nongeometric automorphism.

The map ϕ above induces an automorphism ϕ_{ab} of F_{ab} , the abelianization of F . Let $\text{char } \phi = \det(x \cdot 1 - \phi_{ab}) \in \mathbb{Z}[x]$, the characteristic polynomial of ϕ_{ab} . It is monic and has constant term ± 1 . Stallings' result (which is proved but not stated explicitly in [St]) is

THEOREM. *If for any pair (λ, μ) of roots of $\text{char } \phi$ and integer $r > 0$ one has $(\lambda\mu)^r \neq 1$, then ϕ is not geometric.*

We shall show that Stallings' condition $(\lambda\mu)^r \neq 1$ is generic if $\text{rank } F \geq 3$, in the sense that it defines an open dense subset of the polynomials $p(x) \in \mathbb{Z}[x]$ which are monic and have $p(0) = \pm 1$. The geometric automorphisms are *rare* (Bourbaki's terminology) in the sense that their characteristic polynomials form a nowhere dense subset (Anglo-Saxon terminology [D, p. 621]), when $\text{rank } F \geq 3$.

The case $\text{rank } F = 2$ must be excluded since it is known that any automorphism ϕ of a free group of rank 2 is geometric (compare e.g. [L-S, p. 25, Proposition 4.5]). Note that the eigenvalues λ, μ of $\text{char } \phi$ satisfy $(\lambda\mu)^2 = 1$, so Stallings' necessary condition to be geometric is automatically satisfied here.

The PV automorphisms introduced in [St] to give explicit examples of nongeometric automorphisms enter here also although through the back door, so to speak. One uses the existence of certain PV numbers to show that Stallings' condition $(\lambda\mu)^r \neq 1$ is nonvacuous. It can be satisfied in every degree ≥ 3 by a suitable monic polynomial $p(x) \in \mathbb{Z}[x]$ with $p(0) = -1$.

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2. Stallings' condition. Let $\mathfrak{P}_n$ denote the set of polynomials $p(x) \in \mathbf{Z}[x]$ which are monic and have $p(0) = \pm 1$. Denote by $\mathfrak{P}_n(Q)$ the polynomials in $Q[x]$ which are monic and have constant term ± 1 . We observe that $\mathfrak{P}_n(Q) \cong Q^{n-1} \times \{\pm 1\} \subset Q^n$, affine space over Q of dimension n , where the isomorphism is given by

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_n \mapsto (a_1, a_2, \dots, a_n) \quad (a_n = \pm 1).$$

Topologize $\mathfrak{P}_n(Q)$ by means of this bijection as a subspace of the Zariski topology on Q^n . Recall that a basis for the open sets of this topology is the collection $D(f)$, where $f \in Q[x_1, \dots, x_n]$, and $D(f) = \{(a_1, \dots, a_n) \in Q^n \mid f(a_1, \dots, a_n) \neq 0\}$.

Observe that each of the two connected components $\mathfrak{P}_n^\varepsilon(Q) \cong Q^{n-1} \times \{\varepsilon\}$, $\varepsilon = \pm 1$, is isomorphic with Q^{n-1} . Let $\mathfrak{P}_n^\varepsilon = \mathfrak{P}_n^\varepsilon(Q) \cap \mathfrak{P}_n$.

Suppose now that $p \in \mathfrak{P}_n(Q)$, $p = x^n + a_1 x^{n-1} + \cdots + a_n$. We may factor p in $\overline{Q}[x]$ as $p = \prod_{i=1}^n (x - \lambda_i)$. For each integer $r \geq 1$ define

$$p_r(x) =: \prod_{i=1}^n (x - \lambda_i^r) = x^n + a_1(r) x^{n-1} + \cdots + a_n(r).$$

LEMMA 2.1. $p_r(x) \in Q[x]$ and $a_n(r) = \pm 1$. In addition if $p(x) \in \mathfrak{P}_n$, then $p_r(x) \in \mathfrak{P}_n$.

PROOF. The $a_j(r)$ are elementary symmetric functions of the $\lambda_1^r, \dots, \lambda_n^r$, so lie in Q . In addition they are algebraic integers if $a_i \in \mathbf{Z}$, whence $a_i(r) \in \mathbf{Z}$ in this case. Finally $a_n(r) = (-1)^n (\lambda_1 \cdots \lambda_n)^r = (-1)^{nr+n} ((-1)^n \lambda_1 \cdots \lambda_n)^r = (-1)^{nr+n} a_n^r = \pm 1$.

If $p \in \mathfrak{P}_n(Q)$, let $\tilde{p} = p(0) \cdot x^n \cdot p(1/x)$. It is clear that $\tilde{p} \in \mathfrak{P}_n(Q)$. In addition the operation $p \mapsto \tilde{p}$ preserves $\mathfrak{P}_n$.

LEMMA 2.2. A necessary and sufficient condition that $p \in \mathfrak{P}_n(Q)$ have a pair of roots (λ, μ) with $\lambda\mu = 1$ is that $R(p, \tilde{p}) = 0$, where $R(f, g)$ is the resultant of two polynomials f and g .

PROOF. See [W, §27].

COROLLARY 2.3. A polynomial $p \in \mathfrak{P}_n(Q)$ has a pair of roots (λ, μ) with $(\lambda\mu)^r = 1$ for some $r \geq 1$ iff for some $r \geq 1$, $R(p_r, \tilde{p}_r) = 0$.

Observe that by 2.1 and the theory of symmetric functions $R(p_r, \tilde{p}_r)$ is a polynomial function of the coefficients $a_1, a_2, \dots, a_n$ of p .

We can now state our main result.

THEOREM 2.4. There exists an $N = N(n)$ such that $p \in \mathfrak{P}_n(Q)$ has a pair of roots (λ, μ) satisfying $(\lambda\mu)^r = 1$ for some $r \geq 1$ if and only if $\prod_{i=1}^N R(p_i, \tilde{p}_i) = 0$.

PROOF. We recall that $\lim_{n \rightarrow \infty} \phi(n) = \infty$ where $\phi(n)$ is Euler's function, the number of units in $\mathbf{Z}_n$. (Here is a clever argument due to Fletcher Gross. $\phi(n) \geq \pi(n) - \pi(n/2) - 1$ since a prime number p satisfying $n/2 < p < n$ must be relatively prime to n . The desired limit follows from the prime number theorem.) It follows that there exists $N = N(n)$ such that $\phi(m) > n!$ for $m > N$.

If $(\lambda\mu)^r = 1$, then $\lambda\mu$ is an r th root of unity. Since λ, μ lie in a splitting field of $p \in Q[x]$, the r th root of unity $\lambda\mu$ must be of degree $\leq n!$ over Q . If the order of $\lambda\mu$

is precisely r_1 (so $\lambda\mu$ is a primitive r_1 th root of unity) then $\phi(r_1) \leq n!$. Thus (λ^r, μ^r) is an inverse pair of roots of p_{r_1} and $r_1 \leq N$. Thus $R(p_{r_1}, \bar{p}_{r_1}) = 0$ with $r_1 \leq N$. This completes the proof of 2.4.

As an example, if $n = 3$, then $N = N(3) = 18$, which is impractically large.

COROLLARY 2.5. *The Stallings' condition " $(\lambda\mu)^r = 1$ for some $r \geq 1$ " defines a Zariski closed subset Z of $\mathfrak{P}_n(Q)$ and hence a closed subset of $\mathfrak{P}_n$ in the subspace topology.*

Corollary 2.5 leaves open the question whether or not $Z \cap \mathfrak{P}_n^\epsilon$ is a proper closed subset of $\mathfrak{P}_n^\epsilon$. Indeed, if $n = 2$, $Z \cap \mathfrak{P}_n^\epsilon = \mathfrak{P}_n^\epsilon$ as we observed in the introduction.

3. PV numbers. A *PV number* is an algebraic integer θ such that $|\theta| > 1$ but all the other conjugates of θ lie in the interior of the unit circle. The following properties of *PV numbers* are immediate.

3.1. If θ is *PV*, so is θ^r , $r \geq 1$.

3.2. If $p(x)$ is the monic minimal polynomial satisfied by the *PV number* θ then $p(x) \in \mathbb{Z}[x]$. If θ is a unit (in the algebraic integers) then $p(0) = \pm 1$.

3.3. If the degree over Q of the *PV number* θ is at least 3, then $(\lambda\mu)^r \neq 1$ for any $r \geq 1$. Here (λ, μ) is any pair of roots of the minimal polynomial p of θ .

LEMMA 3.4. *If for each $n \geq 3$ there is a *PV number* θ_n of degree n over Q such that for even n the norm of θ_n is -1 , then $Z \cap \mathfrak{P}_n^\epsilon(Q)$ is a proper closed subset of $\mathfrak{P}_n^\epsilon(Q)$, in the notation of 2.5 ($\epsilon = \pm 1$).*

PROOF. Let the conjugates of θ_n be $\theta_n = \alpha_1, \alpha_2, \dots, \alpha_n$ and let $p(x) \in \mathbb{Z}[x]$ be the minimal polynomial of θ_n . If n is odd, then $p(x)$ and $(-1)^n p(-x)$ are in $\mathbb{Z}[x]$, are monic, and are such that one of them has constant term -1 and the other has constant term $+1$. Since the roots of the second are $-\alpha_1, \dots, -\alpha_n$, it follows from 3.3 that $(\lambda\mu)^r \neq 1$, $r \geq 1$, for any pair of roots either of $p(x)$ or of $(-1)^n p(-x)$. Thus $\mathfrak{P}_n^\epsilon(Q) \cap Z \neq \mathfrak{P}_n^\epsilon(Q)$ for $\epsilon = \pm 1$ if n is odd.

On the other hand if n is even and θ_n has norm -1 , then the constant term of $p(x)$ is -1 , so 3.3 shows that $Z \cap \mathfrak{P}_n^{-1}(Q) \neq \mathfrak{P}_n^{-1}(Q)$.

Next consider θ_n^2 with conjugates $\alpha_1^2, \alpha_2^2, \dots, \alpha_n^2$. By 3.1, θ_n^2 is also *PV* and its norm is 1. Thus the constant term of the minimal polynomial of θ_n^2 , $p_2(x) = \prod_{i=1}^n (x - \alpha_i^2)$, is 1. Again 3.3 shows that $Z \cap \mathfrak{P}_n^{+1}(Q) \neq \mathfrak{P}_n^{+1}(Q)$.

To be able to apply 3.4, we need a supply of *PV numbers*.

THEOREM [Si]. *For each $n \geq 1$ the polynomial*

$$P_n(x) = x^n(x^2 - x - 1) + (x^2 - 1)$$

*has a root θ_{n+2} which is a *PV number* of degree $n + 2$.*

Observe that for n even, the norm of θ_n is -1 , so the numbers θ_n satisfy the hypothesis of 3.4.

Hence we deduce

THEOREM 3.5. *For each $n \geq 3$, $Z \cap \mathfrak{P}_n^\epsilon(Q)$ is a proper closed subset of the affine space $\mathfrak{P}_n^\epsilon(Q)$.*

Next we apply these results to $\mathfrak{P}_n$.

THEOREM 3.6. $Z \cap \mathfrak{P}_n^\epsilon$ is nowhere dense in $\mathfrak{P}_n^\epsilon$.

This result is a consequence of two properties of affine space.

LEMMA 3.7. If U is any nonempty open subset of Q^n , then U contains a point of $\mathbf{Z}^n$.

PROOF. We may assume $U = D(f)$, $f \in Q[x_1, \dots, x_n]$. A standard argument using the fact that $\mathbf{Z}$ is infinite, [Z-S, p. 38], shows that there exist $a_1, a_2, \dots, a_n \in \mathbf{Z}$ with $f(a_1, a_2, \dots, a_n) \neq 0$. Thus $(a_1, a_2, \dots, a_n) \in U \cap \mathbf{Z}^n$.

LEMMA 3.8. Let Z be a proper closed subset of Q^n . Let f be a nontrivial polynomial in $Q[x_1, \dots, x_n]$. Then $Df \cap \mathbf{Z}^n \cap (Q^n - Z) \neq \emptyset$.

PROOF. Df and $Q^n - Z$ are nonempty open subsets of Q^n . Irreducibility of Q^n implies that $Df \cap (Q^n - Z) \neq \emptyset$. Lemma 3.7 shows that $Df \cap (Q^n - Z)$ contains a point with integral coordinates.

PROOF OF 3.6. Since $Z \cap \mathfrak{P}_n^\epsilon(Q)$ is a proper closed subset of $\mathfrak{P}_n^\epsilon(Q) \cong Q^{n-1}$, for any nontrivial polynomial f in $Q[x_1, \dots, x_{n-1}]$, $D(f) \cap (\mathfrak{P}_n^\epsilon(Q) - \mathfrak{P}_n^\epsilon(Q) \cap Z)$ contains an integral point by 3.8. That is, $Df \cap (\mathfrak{P}_n^\epsilon - \mathfrak{P}_n^\epsilon \cap Z) \neq \emptyset$. Thus $Z \cap \mathfrak{P}_n^\epsilon$ is nowhere dense in $\mathfrak{P}_n^\epsilon$. This complete the proof.

THEOREM 3.9. Let $S \subset \mathfrak{P}_n$ denote the set $\{\text{char } \phi\}$ where ϕ ranges over geometric automorphisms of a free group of rank n . Then S is nowhere dense in $\mathfrak{P}_n$ for $n \geq 3$.

PROOF. By Stallings' theorem quoted in the introduction, if $p \in S$, then some pair of roots (λ, μ) of p satisfy $(\lambda\mu)^r = 1$ for some integer $r \geq 1$. Corollary 2.5 shows that $p \in Z$. Thus $S \subset Z \cap \mathfrak{P}_n$. By Theorem 3.6 $Z \cap \mathfrak{P}_n$ is nowhere dense in $\mathfrak{P}_n$ for $n \geq 3$. It follows that S is nowhere dense. This completes the proof.

4. Remarks. In [St] it is conjectured that the fixed point set of a PV automorphism ϕ of a free group F is trivial, where ϕ is called PV if $\text{char } \phi$ has a root θ which is a PV number of degree equal to rank F . Here is a counterexample to that conjecture as stated. Let $\phi: F(x, y) \rightarrow F(x, y)$ be given by $\phi(x) = x^2y$, $\phi(y) = xy$. Then $\text{char } \phi = x^2 - 3x + 1$ with roots $(3 \pm \sqrt{5})/2$. Hence ϕ is PV . However $\phi([x, y^{-1}]) = [x, y^{-1}]$, where $[a, b] = aba^{-1}b^{-1}$.

Of course one could simply patch up Stallings' conjecture by demanding that F be of rank ≥ 3 . However there is a more elegant way of proceeding in light of our results.

We define an automorphism ϕ of a finitely generated free group F to be *generic* if, for any pair of eigenvalues (λ, μ) of $\text{char } \phi$ and integer $r \geq 1$, $(\lambda\mu)^r \neq 1$. By our Theorem 2.4, the generic automorphisms are defined by an open condition in the Zariski topology. We observe also that if ϕ is generic, so is ϕ^r , $r \neq 0$, and $\psi\phi\psi^{-1}$, where $\psi \in \text{Aut}(F)$. Thus if ϕ is generic, so is the entire conjugacy class of ϕ . Observe also that a PV automorphism of F of rank at least three is generic whereas no automorphism of a free group of rank two is generic.

We should like to inquire then whether a generic automorphism of a free group has only trivial fixed points. There is some evidence for this [Sq, G]. Examples are very hard to come by and difficult to compute.

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