ON THE HYPERBOLIC RIESZ MEANS

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ABSTRACT. We define the hyperbolic Riesz means in $\mathbf{R}^2$ by $H_\lambda f = (m_\lambda \hat{f})^\vee$ where $m_\lambda(\xi_1, \xi_2) = (1 - (\xi_1 \xi_2)^2)_+^\lambda$, $\lambda \geq 0$, and show that H_λ is bounded on $L^p(\mathbf{R}^2)$ for $\frac{4}{3} \leq p \leq 4$ and $\lambda > \frac{1}{2}$ or $1 < p < \infty$ and $\lambda \geq 1$.

Introduction. Several authors [1, 3] have studied the Riesz means for various domains in the plane, using the euclidean distance to the boundary of these domains. In the multipliers m_λ , we use the hyperbolic distance to the characteristic set: $\xi_1 \xi_2 = 0$. Related results will appear in a separate paper.

1. We first estimate the Fourier transform of m_λ . A simple calculation gives

$$\hat{m}_\lambda(x, y) = \int_0^\infty \cos\left(\frac{xy}{u}\right) J_{\lambda+1/2}(u) \frac{du}{u^{3/2+\lambda}}.$$

LEMMA 1. *Let*

$$h_\lambda(t) = \int_0^\infty \cos\left(\frac{t}{u}\right) J_{\lambda+1/2}(u) \frac{du}{u^{3/2+\lambda}};$$

then:

- (1) $h_\lambda(t) = O(\log |t|)$ for $t \sim 0$;
- (2) $h_\lambda(t) = O(1/|t|^{3/4+\lambda/2})$ for $|t| \sim \infty$.

PROOF. This is a consequence, after some simple estimates, of the well-known estimates on Bessel functions.

REMARK. From Lemma 1, we deduce the following consequence: let

$$K_\lambda(x, y) = \frac{1}{\Gamma(\lambda + 1)} (1 - (xy)^2)_+^\lambda.$$

Then the operator

$$\mathcal{K}_\lambda f(x, y) = \int_{\mathbf{R}^2} f(x - s, y - t) K_\lambda(s, t) ds dt$$

is not bounded on $L^2(\mathbf{R}^2)$ for any $\lambda \geq 0$.

For a function ϕ of one variable t we let $\eta_+(\gamma) = \int_0^\infty \phi(t) t^{-i\gamma} dt$ and $\eta_-(\gamma) = \int_0^\infty \phi(-t) t^{-i\gamma} dt$.

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LEMMA 2. If a measurable function $\phi(t)$ satisfies: (1) $\int_{-\infty}^{\infty} \phi(t) dt = 0$, (2) η_+ and η_- are C^1 near $\gamma = 0$, and (3) $|\eta_+(\gamma)| + |\eta_-(\gamma)| \leq C/(1 + |\gamma|)$, then the operator

$$Tf(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - s, y - t) \phi(st) ds dt$$

is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$.

PROOF. We write

$$Tf(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - s, y - t) \psi(st) \frac{ds}{|s|} \frac{dt}{|t|}$$

with $\psi(v) = |v| \phi(v)$. We then use the Mellin transform to write

$$\begin{aligned} \int_0^{\infty} \psi(v) v^{-i\gamma} \frac{dv}{v} &= \int_0^{\infty} \phi(v) v^{-i\gamma} dv = \eta_+(\gamma), \\ \int_0^{\infty} \psi(-v) v^{-i\gamma} \frac{dv}{v} &= \int_0^{\infty} \phi(-v) v^{-i\gamma} dv = \eta_-(\gamma). \end{aligned}$$

Then

$$Tf(x, y) = \iint_{st \geq 0} + \iint_{st \leq 0} = 2 \int_{-\infty}^{\infty} (\eta_+(\gamma) + \eta_-(\gamma)) R_{\gamma} f(x, y) d\gamma$$

where

$$R_{\gamma} f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - s, y - t) \frac{ds}{|s|^{1-i\gamma}} \frac{dt}{|t|^{1-i\gamma}}.$$

We now let $M_{\gamma} f = (|\xi_1|^{-i\gamma} |\xi_2|^{-i\gamma} \hat{f})^v$. Then $M_{\gamma} = C_{\gamma}^{-2} R_{\gamma}$ with

$$C_{\gamma} = \frac{\Gamma((1 + i\gamma)/2)}{\pi^{i\gamma+1/2} \Gamma(-i\gamma/2)}.$$

From [2, Lemma (2.4)] we have

$$\|M_{\gamma}\|_{L^p} \leq (\text{const})(1 + |\gamma|^{2|1/p-1/2|} \log^2 |\gamma|)$$

for $1 < p < \infty$. We then write

$$\begin{aligned} Tf(x, y) &= 2 \int_{-1}^1 (\eta_+(\gamma) + \eta_-(\gamma)) R_{\gamma} f(x, y) d\gamma \\ &\quad + 2 \int_{|\gamma| \geq 1} (\eta_+(\gamma) + \eta_-(\gamma)) R_{\gamma} f(x, y) d\gamma \\ &= \text{I} + \text{II}. \end{aligned}$$

For I, we have

$$\text{I} = 2 \int_{-1}^1 (\eta_+(\gamma) + \eta_-(\gamma)) C_{\gamma}^{-2} M_{\gamma} f(x, y) d\gamma.$$

Now, $\eta_+(0) + \eta_-(0) = \int_{-\infty}^{\infty} \phi(t) dt = 0$ by hypothesis (1). By hypothesis (2) $\eta_+(\gamma) + \eta_-(\gamma) \sim (\text{const})\gamma$ for $\gamma \sim 0$. If we use the fact that $c_\gamma \sim \gamma$ for $\gamma \sim 0$, we may write

$$\begin{aligned} I &\simeq (\text{const}) \int_{-1}^1 \gamma C_\gamma^{-2} M_\gamma f(x, y) d\gamma \\ &= (\text{const}) \int_{-1}^1 M_\gamma f(x, y) \frac{d\gamma}{\gamma} + \text{bounded operator} \\ &= (\text{const}) Uf(x, y) + \text{bounded operator} \end{aligned}$$

where $Uf(x, y) = (\sigma \hat{f})^v(x, y)$ with

$$\sigma(\xi_1, \xi_2) = \int_0^{\log \xi_1 \xi_2} \frac{\sin u}{u} du.$$

The multiplier σ is easily seen to verify $|\xi^\alpha \partial_\xi^\alpha \sigma(\xi)| \leq C_\alpha$ for all $\alpha = (\alpha_1, \alpha_2)$, which shows, using a well-known result [4, Theorem 6'], that U is bounded on $L^p(\mathbf{R}^2)$ for all $1 < p < \infty$.

For II we use hypothesis (3), the estimates on $\|M_\gamma\|_{L^p}$ given above and the fact that $C_\gamma = O(\sqrt{|\gamma|})$ as $|\gamma| \rightarrow \infty$, to get

$$\|II\|_p \leq (\text{const}) \int_{|\gamma| \geq 1} \left(\frac{|\gamma|^{2|1/p - 1/2|}}{|\gamma|^2} \log^2 |\gamma| \right) d\gamma \|f\|_p.$$

The last integral is clearly convergent for all $p > 1$.

THEOREM. *The hyperbolic Riesz mean H_λ is bounded on $L^p(\mathbf{R}^2)$ for all $\lambda > 0$ and $p > 1$ such that $\lambda > |2/p - 1|$.*

PROOF. From Lemma 1, we have $H_\lambda f = h_\lambda * f$ where

$$h_\lambda(t) = \int_0^\infty \cos\left(\frac{t}{u}\right) J_{\lambda+1/2}(u) \frac{du}{u^{3/2+\lambda}}.$$

Condition (1) of Lemma 2 is satisfied since

$$\int_{-\infty}^\infty h_\lambda(t) dt = \frac{1}{2} J_{\lambda+1/2}\left(\frac{1}{u}\right) |u|^\lambda \Big|_{u=0} = 0.$$

Condition (2) is easily verified since $h_\lambda(t) = \log|t| + \phi(t)$ for $t \sim 0$, where ϕ is a smooth function, and

$$h_\lambda(t) = \frac{\exp(i\sqrt{|t|})}{|t|^{3/4+\lambda/2}} \psi(t)$$

for $t \sim \infty$, where ψ is a smooth function which verifies $|\partial_t^k \psi(t)| \leq C_k(1 + |t|)^{-k}$, and the integrals

$$\int_0^1 \log^2 t dt \quad \text{and} \quad \int_1^\infty \exp(i\sqrt{t}) \log t \frac{dt}{t^{3/4+\lambda/2}}$$

are convergent. For condition (3), a simple estimate shows that

$$|\eta_+(\gamma)| + |\eta_-(\gamma)| \leq C/(1 + |\gamma|)^\lambda.$$

To conclude we need the integral

$$\int_0^\infty \frac{|\gamma|^{2|1/p-1/2|}}{(1+|\gamma|)^{\lambda+1}} \log^2(1+|\gamma|) d\gamma$$

to be convergent, i.e., $\lambda + 1 - |2/p - 1| > 1$.

COROLLARY. H_λ is bounded on $L^p(\mathbf{R}^2)$ for (1) $\lambda > \frac{1}{2}$ and $\frac{4}{3} \leq p \leq 4$; (2) $\lambda \geq 1$ and $1 < p < \infty$.

REMARK. For the Riesz means in $\mathbf{R}^2$, it is known that the critical index is $\frac{1}{2}$ ($= (n-1)/2$). For $\lambda > \frac{1}{2}$, one has trivially both the L^p -estimates for $1 < p < \infty$ and the pointwise convergence associated with the Riesz means. In the case of the operators H_λ , one needs to dissociate these two properties. For the L^p -estimates, $1 < p < \infty$, we conjecture that the critical index is 1 ($= n/2$). This loss of $\frac{1}{2}$ might occur because the hyperbolas have vanishing curvature at infinity. For the corresponding pointwise convergence, it is an open problem for any $\lambda \geq 0$.

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