RELATIVE C*-SUMS

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ABSTRACT. A "bounded choice" theorem is proved for AW*-subalgebras of a C*-sum.

A classical tool in the structure theory of AW*-algebras is the C*-sum: If A is an AW*-algebra and (h_i) is an orthogonal family of central projections with supremum 1, then A is *-isomorphic to $\oplus h_i A$ (the C*-algebra of all norm-bounded families $x = (x_i)$ with $x_i \in h_i A$, $\|x\| = \sup \|x_i\|$, $x^* = (x_i^*)$ and the coordinatewise algebra operations) [5, Lemma 2.5]. The following generalization has useful applications in the structure theory [2] and embedding theory [3] of AW*-algebras, the idea being that one can simultaneously decompose any AW*-subalgebra B of A (even if B does not contain the h_i):

THEOREM. *Let A be an AW*-algebra, $(h_i)_{i \in I}$ an orthogonal family of central projections of A with $\sup h_i = 1$, B an AW*-subalgebra of A , e the unity element of B . Then:*

- (1) *For each $i \in I$, $h_i B$ is an AW*-subalgebra of A ;*
- (2) *$B \subset \oplus h_i B \subset A$, the inclusions being AW*-embeddings;*
- (3) *$\oplus h_i B$ is the set of all families $(h_i b_i)_{i \in I}$ with $b_i \in B$ and $\sup \|b_i\| < \infty$;*
- (4) *$B = \oplus h_i B$ if and only if $h_i e \in B$ for all $i \in I$.*

PROOF. The hypothesis on B is that it is a norm-closed *-subalgebra of A such that (i) B contains the supremum (as calculated in A) of every orthogonal family of projections in B , and (ii) $b \in B$ implies $\text{RP}(b) \in B$ (where RP denotes right-projection as calculated in A). It follows that (iii) B is itself an AW*-algebra, and the word "orthogonal" can be omitted in (i). {For a closed *-subalgebra B of A , the conditions (i), (ii) are equivalent to the conditions (i), (iii) [6, Lemma 2].}

(1) Let h be any central projection of A . The mapping $B \rightarrow hB$ defined by $b \mapsto hb$ is a *-homomorphism, so its range hB is a C*-algebra [4, 1.8.3]. If $x \in hB$, say $x = hb$ with $b \in B$, then $\text{RP}(x) = h\text{RP}(b)$; by hypothesis $\text{RP}(b) \in B$, so $\text{RP}(x) \in hB$. In particular, every projection $g \in hB$ may be written $g = hf$ with f a projection of B . If (f_α) is a family of projections of B and if $f = \sup f_\alpha$ (as calculated in B or in A —it is the same), then $\sup(hf_\alpha) = hf$; this shows that hB contains the supremum (in A) of every family of its projections. Thus hB is an AW*-subalgebra of A .

(2) The inclusions are obvious (on identifying A with $\oplus h_i A$). Since $h_i B$ is an AW*-subalgebra of A , hence of $h_i A$, it follows that $\oplus h_i B$ is an AW*-subalgebra of

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$\oplus h_i A = A$. Then B , being an AW*-subalgebra of A , is also an AW*-subalgebra of $\oplus h_i B$.

(3) Write $\bar{B} = \oplus B_i$, where $B_i = B$ for all $i \in I$. The mapping $\varphi: \bar{B} \rightarrow \oplus h_i B$ defined by $\varphi((b_i)) = (h_i b_i)$ is a *-homomorphism, therefore its range $\varphi(\bar{B})$ is a C*-subalgebra of $\oplus h_i B$. Let us show that $\varphi(\bar{B})$ contains every projection e of $\oplus h_i B$. Say $e = (h_i b_i)$, where $b_i \in B$ and $\sup \|h_i b_i\| < \infty$. The elements $h_i b_i$ are themselves projections, therefore $h_i b_i = \text{RP}(h_i b_i) = h_i \text{RP}(b_i)$, where $\text{RP}(b_i) \in B$; thus $e = (h_i f_i)$, where the f_i are projections in B , consequently $(f_i) \in \bar{B}$ and $\varphi((f_i)) = e$. Since the AW*-algebra $\oplus h_i B$ is generated by its projections, it follows that $\varphi(\bar{B}) = \oplus h_i B$, which proves (3).

(4) If $h_i e \in B$ for all $i \in I$ then $(h_i e)$ is a central partition of the unity element in the AW*-algebra B , therefore $B = \oplus h_i B$. Conversely, if $B = \oplus h_i B$ then $h_i e \in h_i B \subset B$ for all $i \in I$. $\square$

Property (3) seems striking when expressed in elementary terms: If a is an element of A such that, for every $i \in I$, there exists $b_i \in B$ with $h_i a = h_i b_i$, then the family $(b_i)_{i \in I}$ can be chosen so that $\sup \|b_i\| < \infty$.

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