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AN IMPROVED ESTIMATE IN THE METHOD OF FREEZING

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ABSTRACT. Let $\dot{x} = A(t)x$ and $\lambda_k(t)$ be the eigenvalues of the matrix $A(t)$. The main result of the Method of Freezing [1] states that if $\|A(t)\| \leqslant a$, $\operatorname{Re}\lambda_k(t) \leqslant \lambda_0$ and $\|A(t) - A(s)\| \leqslant \delta|t-s|$, then

$$\|x(t)\| \leqslant \|x(t_0)\| D_\delta \exp(\lambda_0 + 2a\lambda_\delta)(t-t_0) \qquad (t \geqslant t_0)$$

for all solutions of the system, where

$$\lambda_\delta = \left(C_n \cdot \delta/4a^2\right)^{1/(n+1)}.$$

The previous best known value, $C_n = n(n+1)/2$, is reduced to the substantially smaller value $2n^n e^{-n}/(n-1)! < \sqrt{2n/\pi}$.

The main result of the Method of Freezing [1] for linear differential equations can be stated as follows:

Let an n-dimensional system

$$\dot{x} = A(t)x \tag{1}$$

be given and let $\lambda_k(t)$ be the eigenvalues of the matrix $A(t)$. If

$$\|A(t)\| \leqslant a, \tag{2}$$

$$\operatorname{Re}\lambda_k(t) \leqslant \lambda_0, \tag{3}$$

$$\|A(t) - A(s)\| \leqslant \delta|t-s|, \tag{4}$$

then all solutions of the system admit the estimate

$$\|x(t)\| \leqslant \|x(t_0)\| D_\delta e^{(\lambda_0 + 2a\lambda_\delta)(t-t_0)} \qquad (t \geqslant t_0) \tag{5}$$

where

$$\lambda_\delta = \left(C_n \cdot \delta/4a^2\right)^{1/(n+1)}, \qquad C_n = n(n+1)/2,$$

and D_δ depends only on δ.

REMARKS. (i) In the trivial case $\delta = 0$, i.e. $A(t) = \text{const.}$, λ_δ has to be replaced by an arbitrary $\varepsilon > 0$ and D_δ by D_ε.

(ii) If $A(t)$ is differentiable, then (4) is equivalent to $\|\dot{A}(t)\| \leqslant \delta$.

(iii) (5) is true but trivial when $\lambda_\delta \geqslant 1$. So the method is of interest just for δ small, in other words, for systems (1) with "slowly changing" matrix $A(t)$.

We show that for δ small enough, the constant C_n can be replaced by one close to

$$C_n' = 2n^n e^{-n}/(n-1)!. \tag{6}$$

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Since by Stirling's Formula,

$$C_n' = 2n(n^n e^{-n}/n!) < \sqrt{2n/\pi}\,,$$

we have $C_n' < C_n$ for $n = 1, 2, \ldots$ and $C_n' = o(C_n)$.

1. THEOREM 1. *Let* (2)–(4) *hold. Then given* ε, $0 < \varepsilon \leqslant (n+2)^2/2$, *there is* $\delta(\varepsilon) > 0$ *such that for* $\delta < \delta(\varepsilon)$ *estimate* (5) *holds with*

$$\lambda_\delta = \left[(C_n' + \varepsilon)\delta/4a^2\right]^{1/(n+1)}, \qquad C_n' = 2n^n e^{-n}/(n-1)!. \tag{7}$$

The value of $\delta(\varepsilon)$ *can be expressed explicitly*:

$$\delta(\varepsilon) = 4a^2 \cdot \varepsilon^{n+1}\left[2/(n+2)^2\right]^{n+2}. \tag{8}$$

The trivial case $\delta = 0$ *is as in Remark* (i).

To prove this theorem we need a number of preliminary steps.

2. The "frozen" equation. For simplicity we let $t_0 = 0$ in (5); the general case can be treated quite similarly—just replace $(0, t)$ with $(t_0, t_0 + t)$.

Fix a value t_1 ("the point of freezing") and rewrite (1) as

$$\dot{x} = A(t_1)x + \left[A(t) - A(t_1)\right]x.$$

Then by the Variation of Constants Formula, we have for every solution $x(t)$ of (1):

$$x(t) = e^{A(t_1)t}x(0) + \int_0^t e^{A(t_1)(t-s)}\left[A(s) - A(t_1)\right]x(s)\,ds. \tag{9}$$

Notice that this is an *identity in* t_1. Therefore t_1 can be chosen arbitrarily, in particular being a function of t. A proper choice of t_1 will play the crucial role.

3. To estimate the norms in (9) we need the following well-known inequality (e.g. see [**1**] or [**2**]): *If* (2) *and* (3) *hold, then*

$$\|e^{A(t_1)\tau}\| \leqslant p(2a\tau)e^{\lambda_0\tau}, \tag{10}$$

where $p = p_{n-1}$ *and*

$$p_k(z) = 1 + z/1! + \cdots + z^k/k!. \tag{11}$$

Let

$$\|x(t)\| = \|x(0)\|e^{(\lambda_0 + 2a\lambda)t}u(t) \tag{12}$$

and $t_1 = t - y/2a$, where $\lambda > 0$ and y will be chosen later. Then taking norms in (9) and using (4) and (10) we get

$$u(t) \leqslant p(2at)e^{-2a\lambda t} + \delta\int_0^t \left|t - s - \frac{y}{2a}\right| p(2a(t-s))e^{-2a\lambda(t-s)}u(s)\,ds.$$

Now apply a particular case of the general Cone Theorem (e.g. see [**3**]).

Consider an integral inequality

$$u(t) \leqslant f(t) + \int_0^t F(t,s)u(s)\,ds \qquad (t \geqslant 0)$$

where all functions are real valued, continuous and nonnegative. If $f(t)$ is bounded and

$$\int_0^t F(t, s)\, ds \leqslant q < 1 \quad \text{for all } t \geqslant 0,$$

then $u(t)$ is bounded: $u(t) \leqslant \sup f(t)/(1-q)$ $(t \geqslant 0)$.

In our case $f(t) = p(2at)e^{-2a\lambda t}$ is clearly bounded. So if we manage to prove that

$$I = \delta \int_0^t \left| t - s - \frac{y}{2a} \right| \cdot p(2a(t-s))e^{-2a\lambda(t-s)}\, ds \leqslant q < 1$$

for δ and $\lambda = \lambda_\delta$ as in Theorem 1, then $u(t)$ will be bounded and (5) will follow by (12). (The bound for u depends on δ; that is why D_δ appears in (5).)

4. Minimization of the integral. First transform I to a simpler form. Letting $2a(t-s) = r$ and $\delta/4a^2 = \gamma$, we have

$$I = \gamma \int_0^{2at} |r - y| \cdot p(r)e^{-\lambda r}\, dr \leqslant \gamma \int_0^\infty |r-y| \cdot p(r)e^{-\lambda r}\, dr.$$

Now choose y to minimize the integral

(13) $$J = \int_0^\infty |r - y| \cdot p(r)e^{-\lambda r}\, dr = \int_0^y + \int_y^\infty.$$

Setting dJ/dy equal to zero gives the equation in y:

(14) $$\int_0^y p(r)e^{-\lambda r}\, dr = \int_y^\infty p(r)e^{-\lambda r}\, dr,$$

which clearly has a unique positive root y_0. Then (13) yields

$$J_{\min} = -\int_0^{y_0} rp(r)e^{-\lambda r}\, dr + \int_{y_0}^\infty rp(r)e^{-\lambda r}\, dr.$$

The change of variables $z = \lambda y_0$ is convenient, and then the direct evaluation of the above integrals via the elementary formula

$$\int P(x)e^{-\lambda x}\, dx = -e^{-\lambda x}\left[\frac{P(x)}{\lambda} + \frac{P'(x)}{\lambda^2} + \cdots + \frac{P^{(m)}(x)}{\lambda^{m+1}}\right] + C,$$

valid for every mth degree polynomial $P(x)$, shows that (14) takes the form

(15) $$p_\lambda(z) = \tfrac{1}{2}e^z \quad \text{or} \quad 2p_\lambda(z)e^{-z} = 1,$$

where

$$p_\lambda(z) = 1 + \frac{\Lambda_{n-2}}{\Lambda_{n-1}}\frac{z}{1!} + \frac{\Lambda_{n-3}}{\Lambda_{n-1}}\frac{z^2}{2!} + \cdots + \frac{1}{\Lambda_{n-1}}\frac{z^{n-1}}{(n-1)!}$$

and $\Lambda_k = 1 + \lambda + \cdots + \lambda^k$, while

$$J_{\min} = \frac{a_1(z)}{\lambda^2} + \frac{a_2(z)}{\lambda^3} + \cdots + \frac{a_n(z)}{\lambda^{n+1}},$$

where (see (11)) $a_k(z) = k[2p_k(z)e^{-z} - 1]$.

Therefore the only task now is to prove

(16) $$\gamma J_{\min} = \gamma\left[a_1(z)/\lambda^2 + \cdots + a_n(z)/\lambda^{n+1}\right] < 1$$

for $\gamma = \delta/4a^2$ and $\lambda = \lambda_\delta$ as in Theorem 1.

5. Recall that z denotes the only positive root of (15) whose existence has already been established. Also notice that $p_k(z)e^{-z} \leqslant e^z e^{-z} = 1$ and hence $a_k(z) \leqslant k$.

This estimate will suffice for $k \leqslant n-1$, but $a_n(z)$ has to be found more explicitly. We have

$$a_n(z) = n\left[2p_n(z)e^{-z} - 1\right] = n\left[\frac{2z^n e^{-z}}{n!} + 2p_{n-1}(z)e^{-z} - 1\right].$$

Since the function $z^n e^{-z}$ ($z \geqslant 0$) takes on its maximum at $z = n$, we have

$$n \cdot \frac{2z^n e^{-z}}{n!} \leqslant \frac{2n^n e^{-n}}{(n-1)!} = C_n'.$$

Next,

$$\begin{aligned} 2p_{n-1}e^{-z} - 1 &= 2e^{-z}(p_\lambda + p_{n-1} - p_\lambda) - 1 \\ &= 2e^{-z}(p_{n-1} - p_\lambda) \quad (\text{by } (15)) \\ &= \lambda \cdot 2e^{-z}\left[\frac{\lambda^{n-2}}{\Lambda_{n-1}} \cdot \frac{z}{1!} + \frac{\lambda^{n-3} + \lambda^{n-2}}{\Lambda_{n-1}} \cdot \frac{z^2}{2!}\right. \\ &\qquad\qquad \left. + \cdots + \frac{1 + \cdots + \lambda^{n-2}}{\Lambda_{n-1}} \cdot \frac{z^{n-1}}{(n-1)!}\right] \\ &\leqslant \lambda \cdot 2e^{-z}\left[\frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^{n-1}}{(n-1)!}\right] \\ &\leqslant \lambda \cdot 2e^{-z}p_{n-1}(z) \leqslant 2\lambda. \end{aligned}$$

Finally, $a_n(z) = a_n^*(z) + \lambda a_n^{**}(z)$, where

$$a_n^*(z) \leqslant C_n', \qquad a_n^{**}(z) \leqslant 2n,$$

so

$$\begin{aligned} J_{\min} &= \frac{a_1}{\lambda^2} + \cdots + \frac{a_{n-2}}{\lambda^{n-1}} + \frac{a_{n-1} + a_n^{**}}{\lambda^n} + \frac{a_n^*}{\lambda^{n+1}} \\ &\leqslant \frac{1}{\lambda^2} + \cdots + \frac{n-2}{\lambda^{n-1}} + \frac{(n-1) + 2n}{\lambda^n} + \frac{C_n'}{\lambda^{n+1}}. \end{aligned} \tag{17}$$

Notice that

$$C_n' \leqslant \sqrt{2n/\pi} < (n+4)/2 \quad \text{for } n \geqslant 1,$$

hence the sum of the coefficients in the previous line is

$$B = 1 + \cdots + (n-2) + (n-1) + 2n + C_n' < (n+2)^2/2. \tag{18}$$

6. Estimating roots of some polynomials. Consider an equation in λ ($\lambda > 0$):

$$Q_\gamma(\lambda) \equiv \gamma\left[b_1/\lambda^2 + \cdots + b_n/\lambda^{n+1}\right] = 1, \tag{19}$$

or equivalently

$$\lambda^{n+1} = \gamma\left(b_1\lambda^{n-1} + \cdots + b_n\right), \tag{20}$$

where $b_k > 0$, $k = 1, \ldots, n$, and $\gamma > 0$. Let $b_1 + \cdots + b_n = B$.

LEMMA. (i) (19) *has a unique positive root* λ_γ.

(ii) $\lambda_\gamma < \lambda$ *if and only if* $Q_\gamma(\lambda) < 1$.

(iii) *If* $B\gamma < 1$, *then* $\lambda_\gamma < (B\gamma)^{1/(n+1)}$.

(iv) *If* $0 < \varepsilon < B$ *and*

$$\gamma < \gamma(\varepsilon) = \varepsilon^{n+1}/B^{n+2}, \tag{21}$$

then

$$\lambda_\gamma < \left[(b_n + \varepsilon)\gamma\right]^{1/(n+1)}. \tag{22}$$

PROOF. (i) and (ii) are clear because $Q_\gamma(\lambda)$ strictly decreases from ∞ to 0 as λ ranges from 0 to ∞. (iii) If $B\gamma < 1$, then $Q_\gamma(1) = B\gamma < 1$, and by (ii), $\lambda_\gamma < 1$. Then (20) shows that $\lambda_\gamma^{n+1} = \gamma(b_1\lambda_\gamma^n + \cdots + b_n) < B\gamma$. (iv) Given (21) where $0 < \varepsilon < B$, we have $B\gamma < (\varepsilon/B)^{n+1} < 1$, and so by (iii), $\lambda_\gamma < (B\gamma)^{1/(n+1)} < 1$. But then

$$b_1\lambda_\gamma^{n-1} + \cdots + b_{n-1}\lambda_\gamma < (b_1 + \cdots + b_{n-1})\lambda_\gamma < B\lambda_\gamma < B(B\gamma)^{1/(n+1)} < \varepsilon.$$

Now (20) implies (22).

7. Proof of Theorem 1. Look first at the equation (cf. (17))

$$Q_\gamma(\lambda) = \gamma\left[\frac{1}{\lambda^2} + \cdots + \frac{(n-1)+2n}{\lambda^n} + \frac{C_n'}{\lambda^{n+1}}\right]$$

in which, by (18), $B < (n+2)^2/2$. Given $0 < \varepsilon \leqslant (n+2)^2/2$, let

$$\gamma(\varepsilon) = \varepsilon^{n+1}\left[2/(n+2)^2\right]^{n+2} < \varepsilon^{n+1}/B^{n+2}\Big)$$

which is exactly (8). Now fix $0 < \gamma_0 < \gamma(\varepsilon)$ and set

$$\lambda_\delta \equiv \lambda_0 = \left[(C_n' + \varepsilon)\gamma_0\right]^{1/(n+1)},$$

which is just (7). Let z_0 be the root of (15) with this fixed λ_0. Then all $a_k(z_0)$ become fixed, and by (17), $\gamma_0 J_{\min} \leqslant Q_{\gamma_0}(\lambda_0)$.

Look at the equation $Q_{\gamma_0}(\lambda) = 1$. By Lemma (iv), its root is

$$\lambda_{\gamma_0} < \left[(C_n' + \varepsilon)\gamma_0\right]^{1/(n+1)} = \lambda_0,$$

and by Lemma (ii), $Q_{\gamma_0}(\lambda_0) < 1$. So (16) holds, and the proof is completed.

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