LINEAR EXTENSIONS OF CONVEX SUBSETS

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ABSTRACT. We give conditions for linear extensions of closed convex subsets of topological vector spaces to be closed.

1. Introduction. Let A be a nonempty convex subset of a real topological vector space (t.v.s.), and let A^i be the set of all points x in A which belong to the interior of each line segment through x in A . The set A^i is not the interior of A . Obviously A^i is the radial kernel of A in the linear manifold generated by A .

Let $A_f = A \setminus A^i$, and we call A_f the frame of A . The structure of the frame is complicated. Even in a Banach space there may exist a bounded closed convex subset containing more than one point such that $A = A_f$. If a closed convex subset A has the nonempty interior, then A_f coincides with the boundary of A . But the frames of bounded closed convex subsets are, in many cases, dense in themselves.

2. Statement of the theorems. We begin with the following lemmas.

LEMMA 1. *Let A be a nonempty bounded closed convex subset of a Hausdorff t.v.s. Then $A_f \neq \emptyset$ and $A = \text{co } A_f$, where $\text{co } A_f$ denotes the convex hull of A_f .*

PROOF. If A consists simply of one point, then it is clear. Let x be an element of A and $t_0 = \sup\{t > 0 : y + t(x - y) \in A\}$ for y ($y \neq x$) in A . Since the intersection of A and the line L joining two points x and y is compact, we have $y + t_0(x - y) \in A_f$, and hence $A_f \neq \emptyset$. Since x can be written as a convex combination of two extreme points of the line segment $L \cap A$ (both are the elements of A_f), it follows that $x \in \text{co } A_f$. Therefore we obtain $A \subset \text{co } A_f$, and it is clear that $A \supset \text{co } A_f$. The proof is complete.

LEMMA 2. *Let A be a closed convex subset of a t.v.s. with $0 \in A^i$ and $\mathcal{Q}$ the linear extension of A . Then A has nonempty interior in $\mathcal{Q}$ if and only if A_f is closed.*

PROOF. Since the radial kernel of A is the interior of A whenever A has nonempty interior, it is clear that if A has interior points in $\mathcal{Q}$, then A^i is open in $\mathcal{Q}$, hence A_f is closed.

Conversely, if A_f is closed, then since A^i is open in A , there exists a circled neighborhood U_0 of 0 such that $U_0 \cap A \subset A^i$. Let x be an element of $U_0 \cap \mathcal{Q}$. If $x \notin A^i$, then there exists an element x_f of the intersection of A_f and the line segment

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$[0, x]$. Since U_0 is circled, it follows that $[0, x] \subset U_0$ and, hence, $x_f \in U_0 \cap A \subset A^i$. This is a contradiction and hence we obtain $U_0 \cap \mathcal{Q} \subset A^i$. Then A has nonempty interior in $\mathcal{Q}$.

THEOREM 3. *Let A be a closed convex subset of a t.v.s. V with $0 \in A^i$ and let $\mathcal{Q}$ be the linear extension of A . If A_f is closed, then $\mathcal{Q}$ is closed. The converse is true if each closed subspace of V is a barrelled space.*

PROOF. Let $\{x_\alpha; \geq\}$ be a net in $\mathcal{Q}$ which converges to x . If A_f is closed, then from Lemma 2, we can select a neighborhood, U_0 , of 0 such that $U_0 \cap \mathcal{Q} \subset A$. Let V_0 be a circled neighborhood of 0 such that $V_0 + V_0 \subset U_0$. Since $\{x_\alpha\}$ is a Cauchy net, there exists α_0 such that

$$\alpha \geq \alpha_0 \Rightarrow x_\alpha - x_{\alpha_0} \in V_0.$$

Therefore there exists $\lambda_0 > 0$ such that

$$\alpha \geq \alpha_0 \Rightarrow x_\alpha \in x_{\alpha_0} + V_0 \subset \lambda_0 V_0 + V_0 \subset (\lambda_0 + 1)U_0,$$

and this implies

$$\alpha \geq \alpha_0 \Rightarrow x_\alpha \in (\lambda_0 + 1)(U_0 \cap \mathcal{Q}) \subset (\lambda_0 + 1)A.$$

Since $(\lambda_0 + 1)A$ is closed, we obtain $x \in (\lambda_0 + 1)A \subset \mathcal{Q}$, and then $\mathcal{Q}$ is closed.

If the closed subspace $\mathcal{Q}$ is a barrelled space, then A has nonempty interior in $\mathcal{Q}$ because $A \cap (-A)$ is a barrel in $\mathcal{Q}$. Therefore, by Lemma 2, A_f is closed and the proof is complete.

Theorem 3 implies that a linear extension $\mathcal{Q}$ of a compact convex subset A with $0 \in A^i$ of a Banach space is closed if and only if $\mathcal{Q}$ is finite dimensional.

The following corollary is a direct consequence of Theorem 3.

COROLLARY 4. *Let V be a locally convex Hausdorff t.v.s. and $\mathcal{Q}$ a subspace of V' , the dual of V . If there exists a bounded subset M in V such that $M^\circ \cap \mathcal{Q}$ is $\sigma(V', V)$ -closed, then $\mathcal{Q}$ is closed relative to the strong topology, where M° denotes the polar in V' of M . Moreover, if V is semireflexive, then $\mathcal{Q}$ is $\sigma(V', V)$ -closed.*

PROOF. Since M° has nonempty interior with respect to the strong topology, $(M^\circ \cap \mathcal{Q})^i \neq \emptyset$. If $M^\circ \cap \mathcal{Q}$ is $\sigma(V', V)$ -closed, then it is strongly closed, and since it has nonempty interior in $\mathcal{Q}$ with respect to the strong topology, by Theorem 3, $\mathcal{Q}$ is strongly closed.

If V is a semireflexive space, then a strongly closed convex subset is $\sigma(V', V)$ -closed, hence $\mathcal{Q}$ is $\sigma(V', V)$ -closed.

THEOREM 5. *Let V be a Hausdorff barrelled space and A a bounded closed convex subset of V with $0 \in A^i$. If the codimension of the linear extension $\mathcal{Q}$ of A is finite, then $\mathcal{Q}$ is closed.*

PROOF. We prove the theorem for codimension one. The other case can be proved similarly. Let $y \neq 0$ be an element of the algebraic complementary subspace to $\mathcal{Q}$ in V . Let $B = \text{co}(A \cup \{y\})$. Then B is closed. Each z in V can be written as

$z = \alpha y + \lambda a$ ($\alpha \in R$, $\lambda > 0$, $a \in A^i$). Since the line segment $[ta, \beta y]$, for $0 < t < 1$ and $0 < \beta < 1$, is contained by B , we obtain that, for sufficiently small $t > 0$,

$$t_0 y + t(z - t_0 y) = (t_0 + t(\alpha - t_0))y + t\lambda a \in B \quad (0 < t_0 < 1).$$

Therefore, for $0 < t_0 < 1$, $t_0 y$ is an element of the radial kernel of B in V , and hence $B^i \neq \emptyset$. Therefore B has interior points. Let x_0 be an interior point of the line segment $[0, y]$. From Theorem 3, this theorem will follow if it is proved that $(B - x_0) \cap \mathcal{A} \subset A$.

Let $z = \lambda a + (1 - \lambda)y$ ($0 \leq \lambda \leq 1$, $a \in A$) be an element of B . Since x_0 can be expressed as $x_0 = \lambda_0 y$ ($0 < \lambda_0 < 1$), if $z - x_0 \in \mathcal{A}$, then $\lambda = 1 - \lambda_0$. Hence we obtain $z - x_0 = (1 - \lambda_0)a \in A$, and this establishes the desired result.

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