ON THE PRODUCT OF f -PROXIMITIES

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ABSTRACT. A unified theory of products of proximity spaces has been introduced. Under suitable conditions on grill operators f , it has been proved that an arbitrary product of f -proximities is an f -proximity.

1. Introduction. Generalized proximities have been extensively investigated by Lodato [9], Harris [6], Gagrut and Naimpally [5], Sharma and Naimpally [13], Thron and Warren [17, 18] and others. Thron [16] introduced the concept of f -proximities and has shown that for different choices of f one obtains many of the known types of proximities including among others the S -proximities, LO -proximities, RI -proximities, Regular proximities of Harris, classical proximities of Efremovič as well as the basic proximities of Čech. In his dissertation [11] Ori has shown that proximities of Osmatesku and θ -proximities of Fedorčuk can also be subsumed under the scheme of f -proximities.

Products of classical EF -proximities have been investigated by Leader [8]. Relations between products of proximities and products of uniformities have been studied by Stevenson [15]. He studied the phenomenon that two uniformities λ, μ on a set X may induce the same proximity but $\lambda \times \lambda, \mu \times \mu$ may induce different proximities on $X \times X$. Products of Steiner's proximities [14] have been investigated by Hayashi [7].

In this article we propose to make a unified study of the product of proximities belonging to various classes of known types of proximities. Some results of earlier investigators become special cases of our results. Our main result is that if a grill operator f belongs to the class A_1 and the class E_1 (to be defined later) then an arbitrary product of f -proximities is an f -proximity.

2. Preliminaries. In this section we fix our notation, collect several definitions and state some results without proof. In what follows there is always an underlying nonempty set X . It will be convenient to denote elements of X by $x, y, \dots$, its subsets by $A, B, \dots$. Families of subsets will be denoted by $\mathcal{A}, \mathcal{B}, \dots$. In particular, $\mathfrak{F}, \mathfrak{G}$ will be used for filters and grills, respectively. Ultrafilters will be denoted by $\mathcal{U}, \mathcal{V}$. The set of grills and ultrafilters on a set will be denoted by $\Gamma(X)$ and $\Omega(X)$, respectively. For an element $x \in X$, $[A \subset X: x \in A]$ is an ultrafilter. Such an ultrafilter is called a *principal ultrafilter* and is denoted by $\mathcal{U}(x)$. Π will be used to

Received by the editors June 28, 1982 and, in revised form, October 24, 1982.
1980 *Mathematics Subject Classification*. Primary 54A05, 54B10, 54E05.

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0002-9939/83/0000-1450/\$03.00

denote a proximity. Basic results on grills and filters are discussed in [16]. We begin by recalling the definition of a grill.

2.1. DEFINITION. A family $\mathfrak{G}$ of subsets of X will be called a *grill* if it satisfies the following conditions:

$$A \supset B \in \mathfrak{G} \Rightarrow A \in \mathfrak{G}; \quad A \cup B \in \mathfrak{G} \Rightarrow A \in \mathfrak{G} \text{ or } B \in \mathfrak{G}.$$

One of the most important results on grills is that they are unions of ultrafilters. For a function $\psi: X \rightarrow Y$ and for subsets $\mathfrak{A}, \mathfrak{B}$ of $\mathfrak{P}(X), \mathfrak{P}(Y)$ respectively, we define

$$\psi(\mathfrak{A}) = [\psi(A): A \in \mathfrak{A}], \quad \psi^{-1}\{\mathfrak{B}\} = [A \subset X: \psi(A) \in \mathfrak{B}].$$

In particular, if $\mathfrak{G}_1, \mathfrak{G}_2$ are grills on X, Y respectively then $\psi^{-1}\{\mathfrak{G}_2\}$ is a grill on X , $\psi(\mathfrak{G}_1)$ is a grill on $\psi(X)$ and $[B \subset Y: B \supset \psi(A) \text{ for some } A \in \mathfrak{G}_1]$ is a grill on Y .

2.2. DEFINITION. A Čech closure operator (or simply a closure operator) on X is a function $c: \mathfrak{P}(X) \rightarrow \mathfrak{P}(X)$ satisfying

$$\begin{aligned} c(\emptyset) &= \emptyset, \quad A \subset c(A) \quad \text{for all } A \subset X, \\ c(A \cup B) &= c(A) \cup c(B) \quad \text{for all } A, B \subset X. \end{aligned}$$

The pair (X, c) is called a *closure space*. A closure space (X, c) is topological if in addition c satisfies $c(A) \supset c(c(A))$ for all $A \subset X$.

A closure space (X, c) is called T_1 if $c(\{x\}) = \{x\}$ for all $x \in X$. Henceforth the set $\{x\}$ containing the single element x will not be distinguished from the element x .

A function $\psi: (X, c) \rightarrow (Y, k)$ is said to be *continuous at a point* $x \in X$ if $x \in c(A)$ implies $\psi(x) \in k(\psi(A))$. It can easily be verified that this is equivalent to $\psi(\mathfrak{G}_c(x)) \subset \mathfrak{G}_k(\psi(x))$, where $\mathfrak{G}_c(x) = [A \subset X: x \in c(A)]$ is the *adherence grill* of x . ψ is called *continuous on* X if it is continuous at each point of X , that is if $\psi(c(A)) \subset k(\psi(A))$ for all $A \subset X$.

2.3. PRODUCTS OF CLOSURE SPACES. Let $[(X_\alpha, c_\alpha): \alpha \in \Lambda]$ be a family of closure spaces. Let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Define $c: \mathfrak{P}(X) \rightarrow \mathfrak{P}(X)$ by $x \in c(A)$ if and only if for each finite cover $[A_1, A_2, \dots, A_n]$ of A there exists $i, 1 \leq i \leq n$, such that

$$x_\alpha \in c_\alpha(p_\alpha(A_i)) \quad \text{for all } \alpha \in \Lambda,$$

where p_α denotes the projection from X to X_α for all $\alpha \in \Lambda$.

By an argument similar to the one given by Leader [8] it can be proved that c is a closure operator on $\prod X_\alpha$ and it is the largest closure operator on $\prod X_\alpha$ such that each projection is continuous. The pair (X, c) is called the *product space* of the closure spaces $(X_\alpha, c_\alpha), \alpha \in \Lambda$. Moreover if each (X_α, c_α) is a topological space then (X, c) is also a topological space and c is the closure operator corresponding to the product topology.

2.4. BASIC PROXIMITIES. A binary relation Π on $\mathfrak{P}(X)$ is said to be a *basic proximity* on X if the following conditions hold:

$$\Pi = \Pi^{-1}, \quad \Pi(A) = [B \subset X: (A, B) \in \Pi]$$

is a grill containing $\bigcup [\mathfrak{U} \in \mathfrak{N}(X): A \in \mathfrak{U}]$ for all $A \subset X$. Henceforth we shall drop the prefix “basic” and just talk of proximities. The definition of basic proximities given in [2] may easily be seen to be equivalent to our definition. The pair (X, Π) is called a *proximity space* when Π is a proximity on X .

For each subset A of a proximity space (X, Π) we define $c_\Pi(A) = [x \in X: A \in \Pi(x)]$. One can easily verify that c_Π is a closure operator on X satisfying the symmetry axioms (R_0 -axiom), that is if $x \in c(y)$ then $y \in c(x)$. A proximity space (X, Π) is called *separated* if $x \in \Pi(y)$ implies $x = y$. Thus (X, Π) is separated if and only if (X, c_Π) is a T_1 -closure space.

A function $\psi: (X, \Pi_1) \rightarrow (Y, \Pi_2)$ is called *p-continuous* if $A \in \Pi_1(B)$ implies $\psi(A) \in \Pi_2(\psi(B))$. It follows immediately from the definition that if $\psi: (X, \Pi_1) \rightarrow (Y, \Pi_2)$ is *p-continuous* then $\psi: (X, c_{\Pi_1}) \rightarrow (Y, c_{\Pi_2})$ is continuous.

2.5. PRODUCTS OF PROXIMITY SPACES. Let $[(X_\alpha, \Pi_\alpha): \alpha \in \Lambda]$ be a family of proximity spaces. Define a binary relation Π on $\mathcal{P}(X)$, where $X = \prod X_\alpha$, by

$(A, B) \in \Pi$ if and only if for each pair of finite covers

$[A_1, A_2, \dots, A_m]$ of A and $[B_1, B_2, \dots, B_n]$ of B there exist i ($1 \leq i \leq m$) and j ($1 \leq j \leq n$) such that $(p_\alpha(A_i), p_\alpha(B_j)) \in \Pi_\alpha$ for all $\alpha \in \Lambda$.

By an argument similar to the one given by Leader [8] for the product of d -spaces it can be verified that (X, Π) is a proximity space and Π is the largest proximity on X such that each projection p_α is *p-continuous*. Moreover, the closure space (X, c_Π) is the product of the closure spaces $(X_\alpha, c_{\Pi_\alpha})$. The pair (X, Π) is called the *product space* of the proximity spaces (X_α, Π_α) . It can be easily verified that (X, Π) is separated if each space (X_α, Π_α) is separated.

2.6. f -PROXIMITIES. We find it convenient to denote the set on which a proximity Π is defined by $X(\Pi)$ and it is called the *reference set* of Π . f is said to be a *grill operator* if it associates to each pair $(\Pi, \mathcal{G})$, where Π is a proximity and $\mathcal{G}$ is a grill on the same set, a grill $f(\Pi, \mathcal{G})$ on $X(\Pi)$ satisfying the condition,

$$f(\Pi, \mathcal{G}_1) \subset f(\Pi, \mathcal{G}_2)$$

for all proximities Π and all grills $\mathcal{G}_1, \mathcal{G}_2$ on the same set such that $\mathcal{G}_1 \subset \mathcal{G}_2$.

A grill operator f is said to belong to the *class* A_1 ($f \in A_1$) if

$$f(\Pi, \mathcal{G}_1 \cup \mathcal{G}_2) \subset f(\Pi, \mathcal{G}_1) \cup f(\Pi, \mathcal{G}_2)$$

for all proximities Π and all grills $\mathcal{G}_1, \mathcal{G}_2$ on the same set. A proximity Π is said to be an *f-proximity* if

$$f(\Pi, \Pi(A)) \subset \Pi(A) \quad \text{for all } A \subset X(\Pi).$$

f -proximities were introduced by Thron [16]. They were also investigated by Ori [11] and Chattopadhyay [1]. It should be observed that there is a small change in our definition of grill operators, f -proximities and the class A_1 from Thron's original definition in [16].

A large number of particular types of grill operators beyond the six (i, r, b, h, e, s) originally given in [16] are known. They may be found in [11] among others. We list their formulations below except for s .

$$i(\Pi, \mathcal{G}) = \mathcal{G}.$$

$$r(\Pi, \mathcal{G}) = [A \subset X(\Pi): \exists x \in X(\Pi), \mathcal{U}(x) \subset \Pi(A) \cap \mathcal{M}].$$

$$b(\Pi, \mathcal{G}) = [A \subset X(\Pi): c_\Pi(A) \in \mathcal{G}].$$

$$h(\Pi, \mathfrak{G}) = [A \subset X(\Pi): \exists a \in A, \mathfrak{U} \in \Omega(X(\Pi)), \mathfrak{U} \subset \Pi(a) \cap \mathfrak{G}].$$

$$e(\Pi, \mathfrak{G}) = [A \subset X(\Pi): \exists \mathfrak{U} \in \Omega(X(\Pi)), \mathfrak{U} \subset \Pi(A) \cap \mathfrak{G}].$$

We also list the definition of t as given in [11].

$$t(\Pi, \mathfrak{G}) = [A \subset X(\Pi): O_0(\Pi, A) \subset \mathfrak{G}],$$

where

$$O_0(\Pi, A) = [B \subset X(\Pi): B \supset A, c_{\Pi}(X - B) = X - B].$$

It may be noted that the definitions of i , b are identical to and the definition of e is equivalent to those given in [16], but there is a change in the definitions of r and h .

2.7. THEOREM. $[i, r, b, h, e, t] \subset A_1$.

2.8. THEOREM. (1) *EF-proximities are exactly the e-proximities.*

(2) *Riesz proximities are exactly the r-proximities.*

(3) *Lodato proximities are exactly the b-proximities.*

(4) *Regular proximities of Harris are exactly the h-proximities.*

(5) *Basic proximities of Čech are exactly the i-proximities.*

(6) *In [11] it has been proved that on a regular topological space every θ -proximity as defined by Fedorčuk [4] is a t-proximity.*

These results are studied in [11 and 16]. Some of them may also be found in [10].

3. The class E_1 , of grill operators. The classes A_0 , A_1 , A_2 , M and I of grill operators, defined in [16], make no assumption relating the effect of grill operators in different proximity spaces. In order to derive general results on the product of f -proximities some such condition needs to be imposed without which one cannot even prove that the p -isomorphic image of an f -proximity space is an f -proximity space. We begin this section by introducing a class E_1 of grill operators.

3.1. DEFINITIONS. A grill operator f is said to belong to the class E_1 ($f \in E_1$) if the following condition holds:

Given a triple $(\psi, (X, \Pi_1), (Y, \Pi_2))$, where $\psi: (X, \Pi_1) \rightarrow (Y, \Pi_2)$ is p -continuous, then

$$\psi(f(\Pi_1, \psi^{-1}\{\mathfrak{G}\})) \subset f(\Pi_2, \mathfrak{G})$$

for all grills $\mathfrak{G}$ on Y . Note that the definition is meaningful for $\psi^{-1}\{\mathfrak{G}\}$ is a grill on X whenever $\mathfrak{G}$ is a grill on Y .

3.2. THEOREM. $[i, r, b, h, e, t] \subset E_1$.

PROOF. Let (X, Π_1) and (Y, Π_2) be proximity spaces and $\psi: (X, \Pi_1) \rightarrow (Y, \Pi_2)$ be a p -continuous function and $\mathfrak{G}$ be a grill on Y .

Clearly ψ is a continuous function from (X, c_{Π_1}) to (Y, c_{Π_2}) .

(i)

$$\psi(i(\Pi_1, \psi^{-1}\{\mathfrak{G}\})) = \psi(\psi^{-1}\{\mathfrak{G}\}) \subset \mathfrak{G} = i(\Pi_2, \mathfrak{G}).$$

Hence $i \in E_1$.

(ii) For $B \subset X$, $\psi(c_{\Pi_1}(B)) \subset c_{\Pi_2}(\psi(B))$. Hence $\psi(c_{\Pi_1}(B)) \in \mathfrak{G}$ implies $c_{\Pi_2}(\psi(B)) \in \mathfrak{G}$. Consequently

$$\begin{aligned} \psi(b(\Pi_1, \psi^{-1}\{\mathfrak{G}\})) &= \psi([B \subset X: \psi(c_{\Pi_1}(B)) \in \mathfrak{G}]) \\ &\subset [\psi(B): c_{\Pi_2}(\psi(B)) \in \mathfrak{G}] \subset [D \subset Y: c_{\Pi_2}(D) \in \mathfrak{G}] \\ &= b(\Pi_2, \mathfrak{G}). \end{aligned}$$

This shows that $b \in E_1$.

(iii) Let $\mathfrak{U}_0$ be an ultrafilter on X . Then it is easy to verify that $\psi(\mathfrak{U}_0)$ is an ultrafilter on $\psi(X)$ and $\mathfrak{V}_0 = [B \subset Y: B \supset \psi(A) \text{ for some } A \in \mathfrak{U}_0]$ is an ultrafilter on Y . Since ψ is p -continuous it follows that for each $B_0 \subset X$

$$\mathfrak{U}_0 \subset \Pi_1(B_0) \text{ implies } \psi(\mathfrak{U}_0) \subset \Pi_2(\psi(B_0)).$$

Since $\Pi_2(\psi(B_0))$ is a grill, $\mathfrak{V}_0 \subset \Pi_2(\psi(B_0))$. Also $\mathfrak{U}_0 \subset \psi^{-1}\{\mathfrak{G}\}$ implies $\mathfrak{V}_0 \subset \mathfrak{G}$. So we have

$$\begin{aligned} \psi(e(\Pi_1, \psi^{-1}\{\mathfrak{G}\})) &= \psi([B \subset X: \exists \mathfrak{U} \in \Omega(X), \mathfrak{U} \subset \Pi_1(B) \cap \psi^{-1}\{\mathfrak{G}\}]) \\ &\subset [D \subset Y: \exists \mathfrak{V} \in \Omega(Y), \mathfrak{V} \subset \Pi_2(D) \cap \mathfrak{M}] = e(\Pi_2, \mathfrak{G}). \end{aligned}$$

Thus $e \in E_1$.

(iv) By an argument similar to the one given to prove that $e \in E_1$, it can be proved that $r, h \in E_1$.

(v) Let $B \in t(\Pi_1, \psi^{-1}\{\mathfrak{G}\})$. Suppose that $D \subset Y$, $D \supset \psi(B)$ and $c_{\Pi_2}(Y - D) = Y - D$. Then

$$\psi^{-1}(D) \subset X, \quad \psi^{-1}(D) \supset B.$$

Let $x \in c_{\Pi_1}(X - \psi^{-1}(D))$. Then $X - \psi^{-1}(D) \in \Pi_1(x)$. Since $\psi(X - \psi^{-1}(D)) \subset Y - D$ and ψ is p -continuous it follows that $Y - D \in \Pi_2(\psi(x))$. Thus $\psi(x) \in c_{\Pi_2}(Y - D) = Y - D$ and hence $x \notin \psi^{-1}(D)$, consequently $x \in X - \psi^{-1}(D)$. Thus $c_{\Pi_1}(X - \psi^{-1}(D)) = X - \psi^{-1}(D)$ and hence $\psi^{-1}(D) \in \psi^{-1}\{\mathfrak{G}\}$ so that $D \in \mathfrak{G}$. Hence

$$[D \subset Y: D \supset \psi(B), c_{\Pi_2}(Y - D) = Y - D] \subset \mathfrak{G}.$$

Thus $\psi(B) \in t(\Pi_2, \mathfrak{G})$. Hence

$$\psi(t(\Pi_1, \psi^{-1}\{\mathfrak{G}\})) \subset t(\Pi_2, \mathfrak{G}).$$

Thus we have that $t \in E_1$.

4. Products of f -proximities. Now we are in a position to state and prove our main result.

4.1. THEOREM. Let $[(X_\alpha, \Pi_\alpha): \alpha \in \Lambda]$ be a family of f -proximities. Let (X, Π) be the product space of the proximity spaces (X_α, Π_α) , $\alpha \in \Lambda$. Then (X, Π) is an f -proximity space provided $f \in A_1 \cap E_1$.

PROOF. Let D be a subset of $\prod_{\alpha \in \Lambda} X_\alpha$. In order to complete the proof we must show that

$$f(\Pi, \Pi(D)) \subset \Pi(D).$$

Let B be a subset of ΠX_α and $B \in F(\Pi, \Pi(D))$. Suppose that $[B_1, B_2, \dots, B_m]$ and $[D_1, D_2, \dots, D_n]$ are two finite covers of B and D , respectively. Since $B \subset B_1 \cup B_2 \cup \dots \cup B_m$ and $f(\Pi, \Pi(D))$ is a grill on ΠX_α ,

$$B_1 \cup B_2 \cup \dots \cup B_m \in f(\Pi, \Pi(D))$$

and hence there exists i , $1 \leq i \leq m$, such that

$$(4.1.1) \quad B_i \in f(\Pi, \Pi(D)).$$

Since $D \subset D_1 \cup D_2 \cup \dots \cup D_n \subset \Pi X_\alpha$ and Π is a proximity on ΠX_α ,

$$\Pi(D) \subset \Pi\left(\bigcup_{j=1}^n D_j\right) = \bigcup_{j=1}^n \Pi(D_j)$$

consequently, by the definition of grill operators

$$(4.1.2) \quad f(\Pi, \Pi(D)) \subset f\left(\Pi, \bigcup_{j=1}^n \Pi(D_j)\right).$$

Since f belongs to the class A_1 of grill operators,

$$f\left(\Pi, \bigcup_{j=1}^n \Pi(D_j)\right) \subset \bigcup_{j=1}^n f(\Pi, \Pi(D_j)),$$

and hence by (4.1.1) and (4.1.2) it follows that there exists j , $1 \leq j \leq n$, such that

$$(4.1.3) \quad B_i \in f(\Pi, \Pi(D_j)).$$

Observe that if A is a subset ΠX_α and $A \in \Pi(D_j)$ then, since p_α is a p -continuous function from (X, Π) to (X_α, Π_α) for each $\alpha \in \Lambda$,

$$p_\alpha(A) \in \Pi_\alpha(p_\alpha(D_j)) \quad \text{for each } \alpha \in \Lambda$$

and hence

$$\Pi(D_j) \subset p_\alpha^{-1}\{\Pi_\alpha(p_\alpha(D_j))\} \quad \text{for each } \alpha \in \Lambda.$$

Thus by the definition of grill operators,

$$f(\Pi, \Pi(D_j)) \subset f(\Pi, p_\alpha^{-1}\{\Pi_\alpha(p_\alpha(D_j))\}) \quad \text{for each } \alpha \in \Lambda$$

and hence by (4.1.3) it follows that

$$B_i \in f(\Pi, p_\alpha^{-1}\{\Pi_\alpha(p_\alpha(D_j))\}).$$

Consequently,

$$(4.1.4) \quad p_\alpha(B_i) \in p_\alpha(f(\Pi, p_\alpha^{-1}\{\Pi_\alpha(p_\alpha(D_j))\})) \quad \text{for each } \alpha \in \Lambda.$$

Since for each $\alpha \in \Lambda$, p_α is p -continuous and $\Pi_\alpha(p_\alpha(D_j))$ is a grill on X_α and since f belongs to the class E_1 of grill operators,

$$p_\alpha(f(\Pi, p_\alpha^{-1}\{\Pi_\alpha(p_\alpha(D_j))\})) \subset f(\Pi_\alpha, \Pi_\alpha(p_\alpha(D_j))) \quad \text{for each } \alpha \in \Lambda$$

and hence (4.1.4) implies that

$$(4.1.5) \quad p_\alpha(B_i) \in f(\Pi_\alpha, \Pi_\alpha(p_\alpha(D_j))).$$

Since for each $\alpha \in \Lambda$, Π_α is an f -proximity (by the hypothesis of the theorem),

$$f(\Pi_\alpha, \Pi_\alpha(p_\alpha(D_j))) \subset \Pi_\alpha(p_\alpha(D_j)) \quad \text{for each } \alpha \in \Lambda$$

and hence by (4.1.5),

$$p_\alpha(B_i) \in \Pi_\alpha(p_\alpha(D_j)) \quad \text{for each } \alpha \in \Lambda.$$

Thus by the definition of products of proximities we conclude that $B \in \Pi(D)$ and hence $F(\Pi, \Pi(D)) \subset \Pi(D)$.

This completes the proof.

4.2. COROLLARY. *Let $f \in [i, r, b, h, e, t]$. Then a product of f -proximities is an f -proximity.*

PROOF. The result follows by the above theorem and the fact that $[i, r, b, h, e, t] \subset A_1 \cap E_1$ (Theorems 2.7, 3.2).

4.3. REMARK. In view of the above corollary and the results of Theorem 2.8 it follows that a product of proximities belonging to one of the many known classes of proximities belongs to the same class. For example,

- a product of basic proximities is a basic proximity,
- a product of RI -proximities is an RI -proximity,
- a product of LO -proximities is a LO -proximity,

and a product of Harris's proximities is a Harris's proximity etc. In particular, a product of classical EF -proximities is an EF -proximity, which is one of the concluding results of Leader [8].

ACKNOWLEDGEMENT. The author would like to thank the referee for useful comments.

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