PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 1, September 1983

TRANSFERS, CENTERS, AND GROUP COHOMOLOGY[1]

DANIEL H. GOTTLIEB

ABSTRACT. The transfer for fibrations is shown to exist for fibres with finitely generated total integral homology groups. This improvement is applied to cohomology of groups.

1. Introduction. The transfer for Hurewicz fibrations in **[BG]** was constructed for a fibration whose fibre F is homotopy equivalent to a finite CW-complex. In fact, it holds when F has a finitely generated total integral homology group. That is when $H_*(F; \mathbf{Z})$ is finitely generated. A corollary involving ω: $X^X \to X$ also holds for finitely generated $H_*(X; \mathbf{Z})$. These results are stated in the context of (co)homology of groups, where the improvement is particularly useful. There are transfer homomorphisms for surjections $G \xrightarrow{\rho} K$ whose kernels have finitely generated homology. The result concerning ω translates into a theorem about the centers of groups G with finitely generated $H_*(G; \mathbf{Z})$.

We apply these results to obtain the following two theorems. We denote by $[H, G]$ the group generated by commutators $hgh^{-1}g^{-1}$ such that $h \in H$ and $g \in G$, and by $\chi(H)$ the Euler-Poincaré number of the group H. That is

$$\chi(H) = \sum (-1)^i \operatorname{rank}(H_i(H; \mathbf{Z})).$$

THEOREM 4. *Let H be a normal subgroup of G such that $H_*(H; \mathbf{Z})$ is finitely generated. Then $h^{\chi(H)} \in [H, G]$ for all $h \in H \cap [G, G]$.*

THEOREM 5. *Let $H_*(H; \mathbf{Z})$ be finitely generated. Let C be a central subgroup of H. Then $c^{\chi(H)} \in [H, H]$ for all $c \in C$.*

We also apply the transfer to the case of central extensions of free abelian groups of finite rank.

2. Stably finite complexes. Spanier-Whitehead duality is always exposed in the literature as holding for finite complexes. But it is a triviality to observe that it works just as well for stably finite complexes. By a stably finite complex we mean a CW-complex X for which some suspension $S^k X$ is homotopy equivalent to a finite complex.

In fact a duality map is defined by μ: $X \wedge X^* \to S^n$ for some n such that the homomorphism $H_q(X; \mathbf{Z}) \to H^{n-q}(X^*; \mathbf{Z})$ given by the slant product $(\mu^*[S^n]/\cdot)$ is

Received by the editors September 1, 1982.
1980 *Mathematics Subject Classification.* Primary 55R05, 20S05.
[1]This research was supported by a grant from the National Science Foundation.

an isomorphism. Spanier showed that for every finite complex X and some n there is a duality map μ: $X \wedge X^* \to S^n$, and X^* is called the dual of X. It is then immediate that if X is stably finite, there must exist a duality map μ: $X \wedge X^* \to S^n$.

In **[BG]** the concept of Spanier-Whitehead duality is extended to Hurewicz fibrations over B with well-based cross-sections. A *duality map* is a fibre preserving map from $B \times S^n \to E \wedge_B \hat{E}$ (where E and $\hat{E}$ are the total space of fibrations with specified cross-section and $\wedge_B$ means reduced fibrewise join) such that $S^n \to F_b \wedge \hat{F}_b$ is a duality map for every fibre F_b and $\hat{F}_b$ of E and $\hat{E}$ respectively. The existence of duality maps for stably finite complexes then implies that for a Hurewicz fibration E with a finite-dimensional CW-complex B, a well-based cross-section, and fibres stably equivalent to a finite complex there is another such fibration $\hat{E}$ and an n such that

$$B \times S^n \xrightarrow{\mu} E \wedge_B \hat{E}$$

is a duality map. As in the usual theory, there is a duality map

$$\hat{\mu}: E \wedge_B \hat{E} \to B \times S^n.$$

Now if $F \to E \xrightarrow{p} B$ is a Hurewicz fibration, we denote by $\overline{E}$ the fibration E disjoint union with a copy of B, which serves as the cross-section. The transfer map $\tau(f)$ is defined by

$$\begin{array}{c} S^n \times B \xrightarrow{\mu} \overline{E} \wedge_B \hat{E} \xrightarrow{(1,f)\wedge 1} \overline{E} \wedge_B \overline{E} \wedge_B \hat{E} \xrightarrow{1 \wedge \hat{\mu}} \overline{E} \wedge_B (B \times S^n) \\ \hfill \downarrow \quad \\ \hfill (B \times S^n) \wedge_B \overline{E} \end{array}$$

which gives $\tau(f)$: $S^n \wedge (B/A) \to S^n \wedge (E/E_A)$ where $E_A = p^{-1}(A)$ and f: $E \to E$ is a fibre preserving map. All the properties of $\tau(f)$ go over to the case of stably finite fibres without changing a word of the proof.

Now a stably finite CW-complex has a very nice homological characterization. A theorem of Milnor states that a simply connected complex which has finitely generated integral homotopy has the homology type of a finite complex [**W**, Proposition 4.1].

PROPOSITION 1. *A CW-complex X is stably finite if and only if $H_*(X; \mathbf{Z})$ is finitely generated.*

Thus the transfer is defined for fibrations with fibres F such that $H_*(F; \mathbf{Z})$ is finitely generated.

TRANSFER THEOREM. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ a Hurewicz fibration so that $H_*(F; \mathbf{Z})$ is finitely generated, F is a CW-complex, B is a finite-dimensional CW-complex, $A \subset B$ is a subcomplex, and f: $E \to E$ is a map so that $p \circ f = p$. Then there exists an S-map $\tau(f)$: $B/A \to E/E_A$ so that for singular homology $p_* \circ \tau(f)_* = \Lambda_{f'}$ (multiplication by the Lefschetz number of $f' = f | F$: $F \to F$). Also*

(a) *$\tau(f)^* \circ p^* = \Lambda_{f'}$ on singular cohomology.*

(b) *For ring spectra $\tau(f)^*(p^*(\alpha) \cup \beta) = \alpha \cup (\tau(f)^*(\beta))$ and $p_*(\tau(f)_*(x) \cap y) = x \cap \tau(f)^*(y)$.*

A corollary of the transfer is the following result. Let ω: $\Omega B \to F$ be the map arising from the fibration in the above theorem.

ω THEOREM. *$\Lambda_{f'}\{\omega\} = 0 \in \{\Omega B, F\}$ where $\{\ \}$ denotes the stable homotopy classes of maps.*

REMARK. This is the generalization of Theorem 1.1 of [**BG**]. Dold and Puppe can remove the finiteness hypothesis on B in [**DP**; see Theorem 6.2]. Thus B is a CW-complex, not necessarily finite dimensional.

3. Groups. In this section we consider the Transfer Theorem and ω Theorem in the context of group (co)homology.

There is a functor from groups and homomorphisms to spaces and maps so that for every group G there is an Eilenberg-Mac Lane space B_G and for every homomorphism $G \xrightarrow{\rho} G'$ there is a map r: $B_G \to B_{G'}$ which induces the corresponding homomorphism on fundamental group, i.e. r_*: $\pi_1(B_G) \cong G \xrightarrow{\rho_*} G' \cong \pi_1(B_{G'})$.

From now on Γ will always denote a constant group of coefficients. The homology (resp. cohomology) of a group G with coefficients, $H_*(G; \Gamma)$ (resp. $H^*(G; \Gamma)$), is isomorphic to $H_*(B_G; \Gamma)$ (resp. $H^*(B_G; \Gamma)$).

Now a short exact sequence of groups

$$1 \to H \xrightarrow{i} G \xrightarrow{\rho} K \to I$$

gives rise to a fibration of classifying spaces

$$B_H \to B_G \to B_K.$$

If $H_*(H; \mathbf{Z})$ is finitely generated, we may apply the transfer theorem to the fibration.

THEOREM 2. *Let $1 \to H \xrightarrow{i} G \xrightarrow{\rho} K \to 1$ be an exact sequence. Let $H_*(H; \mathbf{Z})$ be finitely generated. Let f: $G \to G$ be a homomorphism such that $\rho f = \rho$. Then,*

(a) *There exists a homomorphism τ_*: $H_*(K; \Gamma) \to H_*(G; \Gamma)$ so that $\rho_* \circ \tau_* = \Lambda_{f'}$, multiplication by the Lefschetz number of $f' = f \mid H$.*

(b) *There is a homomorphism τ^*: $H^*(G, \Gamma) \to H^*(K; \Gamma)$ so that $\tau^*\rho^* = \Lambda_{f'}$.*

(c) *$\tau^*(\rho^*\alpha \cup \beta) = \alpha \cup \tau^*(\beta)$ and $\rho_*(\tau_*(x) \cap y) = x \cap \tau^*(y)$.*

REMARKS. (a) The Lefschetz number

$$\Lambda_{f'} = \sum_{i=0}^{\infty} (-1)^i \left(\operatorname{trace} f_*^{(i)}\right).$$

If f is the identity homomorphism, then $\rho f = \rho$ and $\Lambda_{f'} = \chi(H)$, the Euler-Poincaré number.

(b) There are no conditions on K.

The map ω: $\Omega B \to F$ factors through $F^F, 1 \xrightarrow{\omega_x} F$ where $F^F, 1$ is the space of self maps of F homotopic to the identity and ω_x evaluates f: $F \to F$ at some point $x \in F$. In [**G**, Theorem II.2] if $F = B_G$, then $F^F, 1 = B_C$ where C is the center of G. Also ω_x: $B_C \to B_G$ induces the inclusion of C into F. Thus the ω Theorem gives us the following:

THEOREM 3. *Suppose* $\omega: H \to G$ *is the inclusion homomorphism of a central subgroup* H *into* G. *If* $H_*(G; \mathbf{Z})$ *is finitely generated, then*

$$\chi(G)\omega_* = 0\colon \tilde{H}_*(H; \Gamma) \to \tilde{H}_*(G; \Gamma)$$

and

$$\chi(G)\omega^* = 0\colon \tilde{H}^*(G; \Gamma) \to \tilde{H}^*(H; \Gamma).$$

REMARK. In [**G**, Corollary 4.3], it was shown that if $\chi(G) \neq 0$ and B_G is homotopy equivalent to a finite complex, then G had trivial center. Stallings [**Stg2**] extended this to groups with finite free resolutions. But on the other hand, Baumslag, Dyer and Heller have shown that any abelian group can be the center of a group G such that $\tilde{H}_*(G; \mathbf{Z}) = 0$, [**BDH**, Theorem 7.1].

Now [**KT**] have shown that for every space there is a group which has the same homology. [**BDH**] have shown that one may choose a group G so that if the space were a finite complex then B_G is homotopy equivalent to a finite complex.

4. Applications.

THEOREM 4. *Suppose* $H_*(H; \mathbf{Z})$ *is finitely generated where* H *is a normal subgroup of* G. *If* $H \subset [G, G]$, *then* $h^{\chi(H)} \in [H, G]$ *for all* $h \in H$.

PROOF. Consider the Stallings-Stammbach exact sequence [**Sts**$_1$, Theorem 2.1; **Stch,** c; **Sj**, Theorem 4.4],

$$H_2(G) \xrightarrow{\rho_*} H_2(K) \xrightarrow{\partial} H/[H, G] \xrightarrow{i_*} H_1(G) \xrightarrow{\rho_*} H_1(K) \to 0$$

where $\rho\colon G \to G/H = K$. Now $H_1(G; \mathbf{Z}) = G/[G, G]$. Since $H \subset [G, G]$ we see that $i_* = 0$. Therefore ∂ is onto. Now there is a transfer homomorphism $\tau_*\colon H_2(K) \to H_2(G)$ so that $\rho_*\tau_* = \chi(H)$. So $\chi(H)$ annihiliates every element of $H/[H, G]$.

THEOREM 5. *Suppose* C *is central in* H. *Suppose* $H_*(H; \mathbf{Z})$ *is finitely generated. Then* $c^{\chi(H)} \in [H, H]$ *for all* $c \in C$.

PROOF. Let $\omega\colon C \to H$ be the inclusion. Then

$$\chi(H)\omega_* = 0\colon H_1(C) \cong C \to H/[H, H] \cong H_1(H).$$

REMARK. There is a curious relationship between the above two theorems. Note that C is central in H if and only if C is normal and $[C, H] = 1$. Now for any normal subgroup H of G we have the following series of normal subgroups in H.

$$1 \subset [H, G] \subset H \cap [G, G] \subset H.$$

The first theorem has the hypothesis that $H \cap [G, G] = H$ and the conclusion that $h^{\chi(H)} \in [H, G]$. The second theorem has the hypothesis that $[H, G] = 1$ and the conclusion that $h^{\chi(G)} \in H \cap [G, G]$.

Recall that the lower central series of a group G is a descending sequence of normal subgroups defined by $G_1 = G$, $G_{\alpha+1} = [G_\alpha, G]$ and $G_\beta = \bigcap_{\alpha<\beta} G_\alpha$ for limit ordinals.

COROLLARY 6. *Let* G *be a group so that* $H_*(G_\alpha; \mathbf{Z})$ *is finitely generated and* $\chi(G_\alpha) = \pm 1$. *Then* $G_\alpha = G_\beta$ *for all* $\beta \geqslant \alpha > 1$.

PROOF. We only need to show that $G_\alpha = G_{\alpha+1}$. We apply Theorem 4 for the subgroup $H = G_\alpha$. Then $G_\alpha \subset G_{\alpha+1}$. Hence $G_\alpha = G_{\alpha+1}$.

REMARK. This corollary implies an interesting observation. It is a theorem of Magnus [**M**] that the intersection of the lower central series of a free group is the trivial subgroup 1. So a free group on two generators, F, has the property that $F_\omega = 1$. But if F is the commutator subgroup of some larger group G, then $G_\omega = F$. This follows since $\chi(F) = -1$. There do exist groups G such that $G_2 = F$. For example the knot group of the trefoil knot.

THEOREM 7. *Let $\mathbf{Z}^r \xrightarrow{i} G \xrightarrow{\rho} K$ be a central extension where $\mathbf{Z}^r$ is the free abelian group of rank r. Suppose ϕ: $G \to \mathbf{Z}^r$ is a homomorphism. Then there are transfers τ_*: $H_*(K; \Gamma) \to H_*(G; \Gamma)$ and τ^*: $H^*(G; \Gamma) \to H^*(K; \Gamma)$ so that $\rho_* \circ \tau_* = \det(\phi \circ i)$ and $\tau^* \circ \rho^* = \det(\phi \circ i)$.*

PROOF. Suppose f: $\mathbf{Z}^r \to \mathbf{Z}^r$ is a homomorphism. Now $H_*(\mathbf{Z}^r; \mathbf{Z}) \cong \Lambda(\mathbf{Z}^r)$, the exterior algebra on $\mathbf{Z}^r$. It is not difficult to see that the Lefschetz number of f is $\Lambda_f = \det(I - f)$ where I is the identity.

Now consider ϕ: $G \to \mathbf{Z}^r$. We define Φ: $G \to G$ by $\Phi(g) = g \cdot (\phi(g))^{-1}$. Since $\mathbf{Z}^r$ is in the center of G, Φ must be a homomorphism. Also note that $\rho \circ \Phi = \rho$ since $\phi(g)$ is in the kernel of ρ for all g. Thus there exists transfers for homology and cohomology where the relevant number is the Lefschetz number of $\Phi \mid \mathbf{Z}^r$: $\mathbf{Z}^r \to \mathbf{Z}^r$. But $\Phi \mid \mathbf{Z}^r = I - \phi \circ i$ where we use additive notation. So

$$\Lambda_{\Phi|\mathbf{Z}^r} = \det(I - \Phi \mid \mathbf{Z}^r) = \det(I - (I - \phi \circ i)) = \det(\phi \circ i).$$

Theorem 2 gives the required transfer.

Central extensions are well understood. If $C \to G \to K$ is a central extension, it corresponds to an element $k_G \in H^2(K; C)$. Conversely every element of $H^2(K; C)$ corresponds to a central extension of C by K. There is a cohomology exact sequence

$$\begin{array}{l} 0 \to H^1(K; C) \xrightarrow{\rho^*} H^1(G; C) \xrightarrow{i^*} H^1(C; C) \xrightarrow{\delta} H^2(K; C) \\ \hphantom{0 \to H^1(K; C) \xrightarrow{\rho^*} H^1(G; C) \xrightarrow{i^*} H^1(C; C) \xrightarrow{\delta}} \qquad \downarrow \rho^* \\ \hphantom{0 \to H^1(K; C) \xrightarrow{\rho^*} H^1(G; C) \xrightarrow{i^*} H^1(C; C) \xrightarrow{\delta}} H^2(G; C) \end{array}$$

Now $k_G = \delta(1)$ where

$$1 \in H^1(C; C) \cong \operatorname{Hom}(H_1(C; \mathbf{Z}); C) = \operatorname{Hom}(C; C)$$

corresponds to the identity homomorphism 1: $C \to C$. In the case where $C = \mathbf{Z}^r$ we have the following result.

THEOREM 8. *Let $k \in H^2(K; \mathbf{Z}^r)$ have order N. Then the corresponding central extension $\mathbf{Z}^r \xrightarrow{i} G \xrightarrow{\rho} K$ admits transfers τ_* and τ^* such that $\rho_*\tau_* = N^r$ and $\tau^*\rho^* = N^r$. If k has infinite order then the only possible transfers in integral cohomology satisfy $\tau^*\rho^* = 0$.*

PROOF. Now $\rho^*(k) = 0$, so if k has infinite order and τ^* is a transfer so that $\tau^*\rho^* = N$, then $Nk = \tau^*\rho^*(k) = 0$. Hence $N = 0$.

If $Nk = 0$ we have $\delta(N \cdot 1) = 0$. So $N \cdot 1 \in H^1(\mathbf{Z}^r; \mathbf{Z}^r)$ is in the image of i^*: $H^1(G; \mathbf{Z}^r) \to H^1(\mathbf{Z}^r, \mathbf{Z}^r)$. That implies there is a $\phi' \in \mathrm{Hom}(H_1(G, \mathbf{Z}); \mathbf{Z}^r)$ so that $\phi' \circ i = N \cdot 1$.

Then ϕ: $G \xrightarrow{h} H_1(G; \mathbf{Z}) \xrightarrow{\phi'} \mathbf{Z}^r$ satisfies $\phi \circ i = N$: $\mathbf{Z}^r \to \mathbf{Z}^r$. Here h is the Hurewicz homomorphism which is onto. Now there are transfers τ_* and τ^* so that $\rho_* \tau_* = \det(\phi \circ i) = \det(N) = N^r$ and similarly $\tau^* \rho^* = N^r$.

K. B. Lee has a proof of Theorem 8 which uses cochain arguments and avoids the exact sequence above.

Theorem 8 relates to work of Lawson and Yau. They study compact manifolds of nonpositive curvature in [**LY**]. If M has nonpositive curvature, then M must be a $K(\pi, 1)$. Now Lawson and Yau show that the center of π must be a free abelian group of rank k where k is the dimension of the group of isometries of M. Thus one gets a central extension $0 \to \mathbf{Z}^k \to \pi_1(M) \to K \to 1$ where K is the quotient of $\pi_1(M) = \pi$ and the center $\mathbf{Z}^k$.

Conner and Raymond [**CR**] (see p. 56, paragraph (f)) observe that the classifying element of this extension has finite order. Thus Theorem 8 applies here and there is a transfer for this extension. In particular, for rational coefficients, the homology of M splits as a tensor product of $H^*(T, \mathbf{Q}) \cong \Lambda^k \mathbf{Q}$ and $H^*(K; \mathbf{Q})$.

I would like to thank K. B. Lee for pointing out the connection of Theorem 8 and the above work.

References

[**BDH**] G. Baumslag, E. Dyer and A. Heller, *The topology of discrete groups*, J. Pure Appl. Algebra **16** (1980), 1–47.

[**BG**] J. C. Becker and D. H. Gottlieb, *The transfer for fibrations and duality*, Compositio Math. **33** (1976), 107–133.

[**CR**] P. E. Conner and F. Raymond, *Deforming homotopy equivalences to homeomorphisms in aspherical manifolds*, Bull. Amer. Math. Soc. **83** (1977), 36–85.

[**DP**] A. Dold and D. Puppe, *Duality, trace, transfer* (Proc. Internat. Conf. Geometric Topology), Polish Scientific Publishers, Warsaw, 1980.

[**G**] D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840–856.

[**KT**] D. M. Kan and W. P. Thurston, *Every connected space has the homology of a $K(\pi, 1)$*, Topology **15** (1976), 253–258.

[**LY**] H. Lawson and S.-T. Yau, *Compact manifolds of nonpositive curvature*, J. Differential Geom. **7** (1972), 211–228. MR **48** # 12402

[**M**] W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einen speziellen Ring*, Math. Ann. **111** (1935), 259–280.

[**Sj**] D. Sjerve, *Group actions and generalized periodicity*, Math. Z. **149** (1976), 1–11.

[**Stg1**] J. Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.

[**Stg2**] ______, *Centerless groups—An algebraic formulation of Gottlieb's Theorem*, Topology **4** (1965), 129–134.

[**Stch**] U. Stammach, *Anwendungen der Homologie Theorie der Gruppen auf Zentralreihen und auf Invarianten von Prasentierungen*, Math. Z. **94** (1966), 157–177.

[**W**] C. T. C. Wall, *Finite conditions for CW-complexes*, Ann. of Math. (2) **81** (1965), 56–69.

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907