SQUARE INTEGRABLE REPRESENTATIONS OF UNIMODULAR GROUPS

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ABSTRACT. An elementary proof is given showing that if a continuous irreducible unitary representation of a locally compact unimodular group G has one nontrivial square integrable matrix entry, then all its matrix entries are square integrable. This result was first proved by R. Godement.

In [2], R. Godement proves that if a continuous irreducible unitary representation of a locally compact unimodular group G has one nontrivial square integrable matrix entry, then all its matrix entries are square integrable. His proof uses the theory of functions of positive type. (See [1].)

In this paper, a simple proof of this result is presented, using only elementary functional analysis and Schur's Lemma.

THEOREM 1 (GODEMENT). *Let G be a locally compact, unimodular group, and let ρ be a continuous, irreducible, unitary representation of G on a complex Hilbert space $\mathcal{H}$. Then the following are equivalent:*

- (a) *There exist nonzero vectors $\varphi, \psi \in \mathcal{H}$ such that $\int_G |(\rho(h)\varphi, \psi)|^2 dh$ is finite.*
- (b) *For all $\varphi, \psi \in \mathcal{H}$, $\int_G |(\rho(h)\varphi, \psi)|^2 dh$ is finite.*
- (c) *The representation ρ is equivalent to a subrepresentation of the regular representation of G (on $L^2(G)$).*

THEOREM 2. *If any (hence all) of the conditions in Theorem 1 hold, then there exists a constant λ such that for all $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{H}$,*

$$\int_G (\rho(h)\varphi_1, \psi_1)(\rho(h)\varphi_2, \psi_2) dh = \lambda(\varphi_1, \varphi_2)(\overline{\psi_1}, \overline{\psi_2}).$$

(The bar denotes the complex conjugate $(\overline{\psi_1}, \overline{\psi_2}) = (\psi_2, \psi_1)$.)

REMARK. In [2], Godement proved Theorem 1 only in the case $\varphi_0 = \psi_0$. His proof is reproduced in [3, Chapter XIV]. A different proof can be found in [4, Chapter I, §5.15].

PROOF OF THEOREM 1.

1. (a) $\Rightarrow$ (b). Assume $0 \neq \varphi_0, \psi_0 \in \mathcal{H}$ and $\int_G |(\rho(h)\varphi_0, \psi_0)|^2 dh < \infty$. Let D be the linear span of the vectors $\rho(k)\varphi_0$, $k \in G$. $\mathcal{D}$ is an invariant subspace of $\mathcal{H}$; that is

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$\rho(h)\mathfrak{D} \subseteq \mathfrak{D}$ for all $h \in G$. Thus $\mathfrak{D}$ is dense in $\mathfrak{H}$. By the translation invariance of the measure dh ,

$$\begin{aligned} \int_G |(\rho(h)\rho(k)\varphi_0, \psi_0)|^2 dh &= \int_G |(\rho(hk)\varphi_0, \psi_0)|^2 dh \\ &= \int_G |(\rho(h)\varphi_0, \psi_0)|^2 dh \end{aligned}$$

is finite. Thus $\int_G |(\rho(h)\varphi, \psi_0)|^2 dh$ is finite for all $\varphi \in \mathfrak{D}$.

We may therefore define a new inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{D}$ by

$$(1) \quad \langle \varphi_1, \varphi_2 \rangle = (\varphi_1, \varphi_2) + \int_G (\rho(h)\varphi_1, \psi_0) \overline{(\rho(h)\varphi_2, \psi_0)} dh.$$

We immediately note the following properties of the new inner product:

$$(2) \quad \langle \varphi, \varphi \rangle \geq (\varphi, \varphi),$$

$$\begin{aligned} (3) \quad \langle \rho(k)\varphi_1, \rho(k)\varphi_2 \rangle &= (\rho(k)\varphi_1, \rho(k)\varphi_2) \\ &\quad + \int_G (\rho(hk)\varphi_1, \psi_0) \overline{(\rho(hk)\varphi_2, \psi_0)} dh \\ &= (\varphi_1, \varphi_2) + \int_G (\rho(h)\varphi_1, \psi_0) \overline{(\rho(h)\varphi_2, \psi_0)} dh = \langle \varphi_1, \varphi_2 \rangle. \end{aligned}$$

Let $\mathfrak{H}_1$ be the completion of $\mathfrak{D}$ with respect to the inner product $\langle \cdot, \cdot \rangle$. By (3), $\rho(k)$ extends uniquely to a unitary operator $\rho_1(k)$ on ρ_1 for each $k \in G$.

Let T be the identity map from $\mathfrak{D}$ considered as a subset of $\mathfrak{H}_1$ to $\mathfrak{D}$ considered as a subset of $\mathfrak{H}$. From (2), we see T has norm at most 1. Thus T extends uniquely to a bounded linear operator (which we still denote by T) from $\mathfrak{H}_1$ to $\mathfrak{H}$, $|T| \leq 1$. For $k \in G$, the bounded operators $T\rho_1(k)$ and $\rho(k)T$ agree on the dense set $\mathfrak{D} \subseteq \mathfrak{H}_1$, and are thus equal. The adjoint T^* of T is a contraction from $\mathfrak{H}$ to $\mathfrak{H}_1$. For $k \in G$, we have

$$\rho_1(k)T^* = [T\rho_1(k^{-1})]^* = [\rho(k^{-1})T]^* = T^*\rho(k).$$

Considering next the bounded operator TT^* from $\mathfrak{H}$ to itself, for all k in G we have

$$TT^*\rho(k) = T\rho_1(k)T^* = \rho(k)TT^*.$$

By Schur's Lemma, TT^* is some multiple M of the identity.

The range of T includes $\mathfrak{D}$ and is therefore dense in $\mathfrak{H}$, so T^* is injective. Thus $M \neq 0$, for if $TT^*\varphi$ is zero, then $\|T^*\varphi\| = (TT^*\varphi, \varphi) = 0$, so $\varphi = 0$.

Furthermore, T is also injective. For assume $\varphi \in \mathfrak{H}_1$, $T\varphi = 0$. Then there exists a sequence $\{\varphi_n\}$ in $\mathfrak{D}$ converging in $\mathfrak{H}_1$ to φ . It follows that $\{\varphi_n\} = \{T\varphi_n\}$ converges in $\mathfrak{H}$ to $T\varphi = 0$, and that the functions $(\rho(h)\varphi_n, \psi_0) = (\varphi_n, \rho(h^{-1})\psi_0)$ form a Cauchy sequence in $L^2(G)$. By passing to a subsequence, we may assume $(\varphi_n, \rho(h^{-1})\psi_0)$ converges almost everywhere in G to its L^2 limit. But $(\varphi_n, \rho(h^{-1})\psi_0)$ converges uniformly to zero. Thus $(\varphi_n, \rho(h^{-1})\psi_0)$ converges in $L^2(G)$ to zero, and $\{\varphi_n\}$ converges to $\mathfrak{H}_1$ to 0.

For $\varphi \in \mathfrak{D} \subseteq \mathfrak{H}$, we have $TT^*\varphi = M\varphi = T(M\varphi)$ and hence, because T is injective, we have $T^*\varphi = M\varphi$. Now for $\varphi \in \mathfrak{D} \subseteq \mathfrak{H}_1$, we have $T^*T\varphi = T^*\varphi = M\varphi$; by continuity, this holds for all $\varphi \in \mathfrak{H}_1$.

For $\varphi \in \mathfrak{D}$, we have

$$M \langle \varphi, \varphi \rangle = \langle T^* T \varphi, \varphi \rangle = (T \varphi, T \varphi) = (\varphi, \varphi),$$

which we may rewrite, using (1), as

$$M(\varphi, \varphi) + M \int_G |(\rho(h)\varphi, \psi_0)|^2 dh = (\varphi, \varphi).$$

Substituting φ_0 for φ , we see $0 < M < 1$, for if $M = 1$, then the dense set $\mathfrak{D}$ is orthogonal to the nonzero vector ψ_0 .

We have thus established, for $\varphi \in \mathfrak{D}$, the equality

$$(4) \quad \int_G |(\rho(h)\varphi, \psi_0)|^2 dh = \frac{1-M}{M}(\varphi, \varphi) < \infty.$$

Now consider an arbitrary vector $\varphi \in \mathfrak{H}$. Let $\{\varphi_n\}$ be a sequence in $\mathfrak{D}$ that converges in $\mathfrak{H}$ to φ . By (4), we see that the sequence of functions $(\rho(h)\varphi_n, \psi_0)$ is Cauchy in $L^2(G)$ and converges in $L^2(G)$ to a function with norm

$$\lim_{n \rightarrow \infty} \left[\frac{1-M}{M}(\varphi_n, \varphi_n) \right]^{1/2} = \left[\frac{1-M}{M}(\varphi, \varphi) \right]^{1/2}.$$

By passing to a subsequence, we may assume $(\rho(h)\varphi_n, \psi_0) = (\varphi_n, \rho(h^{-1})\psi_0)$ converges almost everywhere to its L^2 limit. But $(\varphi_n, \rho(h^{-1})\psi_0)$ converges uniformly to $(\varphi, \rho(h^{-1})\psi_0)$. We have now established (4) for all $\varphi \in \mathfrak{H}$.

We have so far proved that if $0 \neq \varphi_0, \psi_0 \in \mathfrak{H}$ and $\int_G |(\rho(h)\varphi_0, \psi_0)|^2 dh$ is finite, then there exists a constant M_{ψ_0} such that $\int_G |(\rho(h)\varphi, \psi_0)|^2 dh = M_{\psi_0}(\varphi, \varphi)$ is finite for all $\varphi \in \mathfrak{H}$.

As

$$\begin{aligned} \int_G |(\rho(h)\varphi, \psi)|^2 dh &= \int_G |(\varphi, \rho(h^{-1})\psi)|^2 dh \\ &= \int_G |(\varphi, \rho(h)\psi)|^2 dh = \int_G |(\rho(h)\psi, \varphi)|^2 dh, \end{aligned}$$

it also follows under the above hypotheses that

$$\int_G |(\rho(h)\varphi_0, \psi)|^2 dh = M_{\varphi_0}(\psi, \psi)$$

is finite for all $\psi \in \mathfrak{H}$. It now follows that $\int_G |(\rho(h)\varphi, \psi)|^2 dh$ is finite for all $\varphi, \psi \in \mathfrak{H}$.

2. (b) $\Rightarrow$ (c). The map $A_\varphi: \mathfrak{H} \rightarrow L^2(G)$ defined by $(A_\varphi\psi)(h) = (\psi, \rho(h)\varphi)$ is easily seen to be an intertwining operator between ρ and the left regular representation π : $[\pi(g)f](h) = f(g^{-1}h)$ on $L^2(G)$.

$$\begin{aligned} [(\pi(g) \circ A_\varphi)\psi](h) &= [\pi(g)(A_\varphi\psi)](h) = (A_\varphi\psi)(g^{-1}h) \\ &= (\psi, \rho(g^{-1}h)\varphi) = (\rho(g)\psi, \rho(h)\varphi) = [A_\varphi(\rho(g)\psi)](h) \\ &= [(A_\varphi \circ \rho(g))\psi](h). \end{aligned}$$

Furthermore, by Theorem 2 (whose proof below requires only condition (b) of Theorem 1), A_φ is bounded. In fact, if φ is chosen to have norm $\lambda^{-1/2}$, then A_φ is an isometry.

3. (c) $\Rightarrow$ (a). The reader is referred to [4, §5.15] for a simple proof of this fact.

PROOF OF THEOREM 2. By the Schwarz inequality, the integral

$$\int_G (\rho(h)\varphi_1, \psi_1) \overline{(\rho(h)\varphi_2, \psi_2)} dh$$

is well defined and bounded in absolute value by $M_{\psi_1}^{1/2} M_{\psi_2}^{1/2} \|\varphi_1\| \|\varphi_2\|$ for all $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{H}$. For fixed ψ_1 and ψ_2 , the integral is a bounded sesquilinear form on $\mathcal{H}$. There is therefore a bounded operator T_{ψ_1, ψ_2} on $\mathcal{H}$, $\|T_{\psi_1, \psi_2}\| \leq M_{\psi_1}^{1/2} M_{\psi_2}^{1/2}$, such that the integral equals $(T_{\psi_1, \psi_2} \varphi_1, \varphi_2)$ for all $\varphi_1, \varphi_2 \in \mathcal{H}$.

From the translation invariance of the measure dh , we have for all $k \in G$, $\varphi_1, \varphi_2 \in \mathcal{H}$

$$\begin{aligned} (T_{\psi_1, \psi_2} \varphi_1, \varphi_2) &= (T_{\psi_1, \psi_2} \rho(k) \varphi_1, \rho(k) \varphi_2) \\ &= (\rho(k)^{-1} T_{\psi_1, \psi_2} \rho(k) \varphi_1, \varphi_2). \end{aligned}$$

By Schur's Lemma, T_{ψ_1, ψ_2} is some multiple M_{ψ_1, ψ_2} of the identity

$$(5) \quad \int_G (\rho(h)\varphi_1, \psi_1) \overline{(\rho(h)\varphi_2, \psi_2)} dh = M_{\psi_1, \psi_2}(\varphi_1, \varphi_2),$$

and

$$\begin{aligned} (6) \quad \int_G (\rho(h)\varphi_1, \psi_1) \overline{(\rho(h)\varphi_2, \psi_2)} dh &= \int_G (\varphi_1, \rho(h^{-1})\psi_1) \overline{(\varphi_2, \rho(h^{-1})\psi_2)} dh \\ &= \int_G (\varphi_1, \rho(h)\psi_1) \overline{(\varphi_2, \rho(h)\psi_2)} dh \\ &= \int_G (\rho(h)\psi_2, \varphi_2) \overline{(\rho(h)\psi_1, \varphi_1)} dh \\ &= M_{\varphi_2, \varphi_1}(\psi_2, \psi_1). \end{aligned}$$

Taking $\varphi_1 = \varphi_2 = e$ a unit vector in (5) and (6), we find $M_{\psi_1, \psi_2} = M_{e, e}(\psi_2, \psi_1)$.

$$\begin{aligned} (7) \quad \int_G (\rho(h)\varphi_1, \psi_1) \overline{(\rho(h)\varphi_2, \psi_2)} dh &= M_{\psi_1, \psi_2}(\varphi_1, \varphi_2) = M_{e, e}(\psi_2, \psi_1)(\varphi_1, \varphi_2) \\ &= M_{e, e}(\varphi_1, \varphi_2) \overline{(\psi_1, \psi_2)}. \end{aligned}$$

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