ON NONISOMORPHIC ROOM SQUARES

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ABSTRACT. Let $NR(s)$ denote the number of nonisomorphic Room squares of side s . We prove that for s sufficiently large, $NR(s) \geq \exp(cs^2)$ for some absolute constant c . More precisely, $NR(s) \geq .19 \exp(.04s^2)$ for $s \geq 153$ odd; and $NR(s) \geq .19 \exp(.09s^2)$ for $s \geq 1001$ odd.

1. Introduction. Let s be a positive odd integer, and let S be a set of size $s + 1$. A *Room square of side s* is an s by s array, R , which satisfies the following properties:

(0) each cell of R either is empty or contains an unordered pair of elements (symbols) of S ,

(1) every symbol occurs in precisely one cell of each row and column of R ,

(2) every unordered pair of symbols occurs in precisely one cell of R .

Room squares have appeared in the literature as early as 1850 (see Kirkman [5]). They have been studied extensively since the 1960s, and the existence question was solved in 1974 by Mullin and Wallis [8]. We state their result as

THEOREM 1.1. *There exists a Room square of side s if and only if s is an odd positive integer other than 3 or 5.*

For $i = 1, 2$, let R_i be a Room square of side s based on the symbol set S_i , and let $\phi: S_1 \rightarrow S_2$ be a bijection. R_1^ϕ is defined to be the Room square based on symbol set S_2 , in which x is replaced by $\phi(x)$ for all $x \in S_1$. We say that R_1 and R_2 are *isomorphic* Room squares if there exists some ϕ such that R_2 can be obtained from R_1^ϕ by permutations of rows and columns. It is clear that isomorphism is an equivalence relation. It is thus natural to ask how many nonisomorphic Room squares exist for each side s . We will denote this quantity by $NR(s)$. We are able to show that $NR(s) \geq \exp(cs^2)$ for some absolute constant c , so that clearly $NR(s)$ grows extremely rapidly.

The *transpose* of a Room square R is the Room square R^T obtained by interchanging the roles of the rows and columns of R . Two Room squares R_1 and R_2 are *equivalent* if R_1 is isomorphic to either of R_2 or R_2^T (this also clearly yields an equivalence relation). The number of inequivalent Room squares will be denoted by $IR(s)$. The following lemma is immediate.

LEMMA 1.2. $IR(s) \geq \frac{1}{2}NR(s)$

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We note that strict inequality will hold if there exists a Room square of side s which is isomorphic to its transpose. However, our lower bounds for $\text{NR}(s)$ provide bounds for $\text{IR}(s)$ as well.

The question of determining the number of nonisomorphic designs of a given type has been studied for several types of designs: Steiner triple systems (see [15]), Steiner quadruple systems [4], and one-factorizations of complete graphs [1 and 7]. In [6], Lindner establishes that $\text{IR}(s)$ is very large for certain s , but our paper provides the first proof that $\text{NR}(s) > 1$ for all but a finite number of sides s .

The number of inequivalent and nonisomorphic Room squares of side 7 was determined exactly in [3 and 14].

2. A recursive construction. We obtain our lower bounds on $\text{NR}(s)$ by using a recursive construction for Room squares based on group divisible designs. A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{Q})$ which satisfies:

(0) X is a finite set (elements of which are called *points*), and $\mathcal{G}$ and $\mathcal{Q}$ are sets of nonempty subsets of X (elements of $\mathcal{G}$ are called *groups* and elements of $\mathcal{Q}$ are called *blocks*),

(1) $\mathcal{G}$ is a partition of X ,

(2) every block has size at least two, and the groups and blocks together contain every unordered pair of points exactly once.

A *transversal design* (or $\text{TD}(k, n)$) is a GDD with $|X| = kn$, having k groups of size n , and n^2 blocks of size k . It is well known that a $\text{TD}(k, n)$ is equivalent to a set of $k - 2$ mutually orthogonal Latin squares of order n .

Our recursive construction starts with a suitable transversal design, replaces each block by an array called a frame, and then replaces each group by a Room square. Let X be a set, and let $\{S_1, S_2, \dots, S_r\}$ be a partition of X . An $\{S_1, S_2, \dots, S_r\}$ -*frame* is a square array F of size $|X|$, with rows and columns indexed by X , which satisfies:

(0) every cell of F either is empty or contains an unordered pair of symbols of X ,

(1) for every i , $1 \leq i \leq r$, the cells of $F(s, s')$, with $\{s, s'\} \subseteq S_i$, are empty (these empty cells are called *holes*),

(2) each symbol of $X \setminus S_i$ occurs in precisely one cell of each row and each column $s \in S_i$ of F ,

(3) the pairs occurring in F are precisely those $\{s, s'\}$ with $(s, s') \in X^2 \setminus \bigcup_{i=1}^r S_i^2$.

Suppose R is a Room square of side s on symbol set $S \cup \{\infty\}$ ($|S| = s$). Index the rows and columns of R so that $\{\infty, S\}$ occurs in cell (s, s) for each $s \in S$. If the contents of these cells are deleted, an $\{\{s\} : s \in S\}$ -frame is formed. Conversely, given such a frame, one can “complete” it to a Room square by filling in the holes (i.e. cells (s, s)), appropriately.

The *type* of an $\{S_1, \dots, S_r\}$ -frame is the multiset $\{|S_i| : 1 \leq i \leq r\}$. We will use the notation $1^{t_1} 2^{t_2} \dots$ to describe the type of frame where there are precisely t_i S_j ’s of size i ($i \geq 1$). Thus the above discussion demonstrates that a Room square of side s is equivalent to a frame of type 1^s .

The following two frames are essential ingredients in our recursive construction.

LEMMA 2.1. *There exist frames of type 1^9 and $1^8 3^1$.*

PROOF. A frame of type 1^9 is equivalent to a Room square of side 9, which exists by Theorem 1.1. The frame of type 1^9 in Figure 1 was given by Beaman and Wallis in [2]. The frame of type $1^8 3^1$ was found by the first author and is presented in Figure 2. Both frames were found by the use of the computer.

Figure 1. A frame of type 1^9

	1	2	3	4	5	6	7	8	9
1			4, 9	3, 7	2, 8		5, 6		
2	8, 9					5, 7	3, 4		1, 6
3		5, 8			6, 9	2, 4		1, 7	
4		3, 6	7, 8			1, 9		2, 5	
5		7, 9		1, 2		3, 8		4, 6	
6	4, 5						1, 8	3, 9	2, 7
7			2, 6	5, 9	1, 3				4, 8
8	6, 7	1, 4					2, 9		3, 5
9	2, 3		1, 5	6, 8	4, 7				

Figure 2. A frame of type $1^8 3^1$

	1	2	3	4	5	6	7	8	9	10	11
1				5, 9			4, 8	7, 11			6, 10
2				7, 10			6, 11	4, 9			5, 8
3					4, 10	5, 11			6, 8	7, 9	
4	7, 8	9, 11					1, 5	2, 6	3, 10		
5	6, 9	8, 10				1, 4				3, 11	2, 7
6	10, 11		8, 9		1, 7			3, 5		2, 4	
7				1, 6	8, 11	9, 10			2, 5		3, 4
8		5, 7	4, 11		3, 6		2, 10				1, 9
9		4, 6	5, 10	2, 11		3, 7				1, 8	
10	4, 5		6, 7			2, 8	3, 9		1, 11		
11				3, 8	2, 9			1, 10	4, 7	5, 6	

Let $(X, \mathcal{G}, \mathcal{Q})$ be a $\text{TD}(9, n)$, where $\mathcal{G} = \{G_i: 1 \leq i \leq 9\}$, and let $W \subseteq G_9$. Denote $t = |W|$. Define a function S , with domain X , by

$$S(x) = \begin{cases} x & \text{if } x \in X \setminus W, \\ \{x_i: 1 \leq i \leq 3\} & \text{if } x \in W. \end{cases}$$

For every $A \in \mathcal{A}$, let F_A be any $\{S(x): x \in A\}$ -frame. (Such a frame is of type 1^9 or $1^8 3^1$.) Then $F = \bigcup_{A \in \mathcal{A}} F_A$ is a $\{\bigcup_{x \in G_i} S(x): 1 \leq i \leq 9\}$ -frame of type $n^8(n+2t)^1$ by [11, Construction 2.2]. Now, we complete F to a Room square of side $9n+2t$. Let $\infty \notin \bigcup_{x \in X} S(x)$, and place a Room square, on symbol set $\{\infty\} \cup (\bigcup_{x \in G_i} S(x))$ in the hole of F induced by $\bigcup_{x \in G_i} S(x)$, for $1 \leq i \leq 9$. The resultant array is a Room square by [12, Theorem 3.1]. We summarize this as

LEMMA 2.2. *Suppose there is a $\text{TD}(9, n)$, with n odd, and $0 \leq t \leq n$. If there exist Room squares of sides n and $n+2t$, then there exists a Room square of side $9n+2t$.*

Our basic method is as follows. We construct a large number of distinct Room squares on a specified symbol set. We can obtain a naive upper bound on the number of Room squares isomorphic to a given Room square. The quotient of these two quantities provides a lower bound on the number of nonisomorphic Room squares. This is done in the next section.

3: A bound. We first prove that there are many distinct frames of the types 1^9 and $1^8 3^1$ (on fixed symbol sets).

LEMMA 3.1. *There are at least $9!$ distinct frames of type 1^9 , on a fixed symbol set.*

PROOF. The frame F_9 of Figure 1 is an $\{\{i\}: 1 \leq i \leq 9\}$ -frame. For Π a permutation of $\{1, 2, \dots, 9\}$, let F_9^Π be the frame defined by $F_9^\Pi(\Pi(i), \Pi(j)) = \{\Pi(s), \Pi(t)\}$ where $F_9(i, j) = \{s, t\}$. It is easily seen that F_9^Π is also a $\{\{i\}: 1 \leq i \leq 9\}$ -frame. We show that the $9!$ permutations of $\{1, 2, \dots, 9\}$ give rise to distinct frames.

Suppose that $F_9^\Pi = F_9^\Phi$. Then, clearly $F_9^{11\Phi^{-1}} = F_9$. Thus, it suffices to show that $F_9^\Pi \neq F_9$ for all Π . This can be established with only a moderate amount of case-work.

If $F(i, j)$ is nonempty, then $F(\Pi(i), \Pi(j))$ must be nonempty. Thus, there are only 36 possibilities for $\{\Pi(i), \Pi(j)\}$, determined by the filled cells of F_9 . We let $i = 1, j = 3$, and consider each case.

All cases are handled similarly. We give an example. Suppose $(\Pi(1), \Pi(3)) = (2, 7)$. Then $\{\Pi(4), \Pi(9)\} = \{3, 4\} = F_9(2, 7)$. If $\Pi(4) = 4$ and $\Pi(9) = 3$, then $F(\Pi(4), \Pi(9)) = F(4, 3)$ must be empty. It is not, so we have a contradiction. Thus, suppose $\Pi(4) = 3$ and $\Pi(9) = 4$. Then $\{3, \Pi(7)\} = \{\Pi(4), \Pi(7)\} = F(\Pi(9), \Pi(5)) = F(4, \Pi(5))$. Since $F(4, 2) = \{3, 6\}$, we have $\Pi(5) = 2$ and $\Pi(7) = 6$. Then $F(\Pi(4), \Pi(7)) = F(3, 6)$ must be empty. It is not, so we have a contradiction in this case as well.

The other cases are handled similarly.

LEMMA 3.2. *There are at least $8!(3!)^3$ distinct frames of type $8^1 3^1$, on a fixed symbol set.*

PROOF. We start with the $\{\{1, 2, 3\}, \{i\}: 4 \leq i \leq 11\}$ -frame F_{11} presented in Figure 2. Let α and β be permutations of $\{1, 2, 3\}$ and let Π be a permutation of $\{1, 2, \dots, 11\}$ such that $\{\Pi(1), \Pi(2), \Pi(3)\} = \{1, 2, 3\}$. Define the frame $F_{11}^{\alpha, \beta, \Pi}$ as follows:

$$\begin{aligned} & \text{for } 4 \leq i, j \leq 11, F_{11}^{\alpha, \beta, \Pi}(\Pi(i), \Pi(j)) = \{\Pi(s), \Pi(t)\}; \\ & \text{for } 1 \leq i \leq 3, 4 \leq j \leq 11, F_{11}^{\alpha, \beta, \Pi}(\alpha(i), \Pi(j)) = \{\Pi(s), \Pi(t)\}; \end{aligned}$$

and for $4 < i < 11$, $1 < j < 3$, $F_{11}^{\alpha, \beta, \Pi}(\Pi(i), \beta(j)) = \{\pi(s), \Pi(t)\}$, where, in each case $F_{11}(i, j) = \{s, t\}$. It is easily shown that each $F_{11}^{\alpha, \beta, \Pi}$ thus constructed is a $\{\{1, 2, 3\}, \{i\}: 4 \leq i \leq 11\}$ -frame. We assert that the $(3!)^3 8!$ frames thus produced are distinct.

Again, it suffices to show that $F_{11}^{\alpha, \beta, \Pi} \neq F_{11}$ for any α, β, Π . Consider $F_{11}(7, 5) = \{8, 11\}$. Then $F_{11}^{\alpha, \beta, \Pi}(\Pi(7), \Pi(5)) = \{(8), \Pi(11)\}$. Now $\Pi(5), \Pi(7), \Pi(8), \Pi(11) \in \{4, 5, \dots, 11\}$ so we see from F_{11} that $\{\Pi(8), \Pi(11)\} = \{8, 11\}, \{9, 10\}, \{4, 7\}$ or $\{5, 6\}$. These four cases are easily disposed of, as in Lemma 3.1, proving the result.

THEOREM 3.3. *Suppose there exists a $\text{TD}(9, n)$, with n odd and $0 \leq t \leq n$. Then there are at least*

$$(9!)^{(n-t)n} \cdot (8! (3!)^3)^{nt} \cdot (2(n!)^2)^8 \cdot 2((n+2t)!)^2$$

distinct Room squares of side $9n + 2t$, on a fixed symbol set.

PROOF. We apply Lemma 2.2. In the construction of the Room square of side $9n + 2t$, there are $(n-t)n$ blocks which miss W , each of which correspond to a frame of type 1^9 . The remaining tn blocks correspond to a frame of type $1^8 3^1$. We thus obtain $(9!)^{(n-t)n} (8! (3!)^3)^{tn}$ distinct frames of type $n^8 (n+2t)^1$.

Now we fill in the holes of this frame. Let R be any Room squares of side n , by permuting rows and columns of R , and transposing (i.e. interchanging the function of rows and columns), we can obtain $2(n!)^2$ distinct Room squares of side n . We fill in eight Room squares of side n and one of side $n + 2t$; so each frame can be completed in at least $(2(n!)^2)^8 2((n+2t)!)^2$ ways.

It is easily seen that all the Room squares obtained by this construction are distinct.

COROLLARY 3.4. *Suppose there exists a $\text{TD}(9, n)$ and $9n \leq s \leq 11$, where n and s are odd. Then*

$$\text{NR}(s) \geq \frac{(9!)^{n^2} 2^9 (n!)^{18}}{((11n)!)^2 (11n+1)!}.$$

PROOF. First, write $s = 9n + 2t$, where $0 \leq t \leq n$. We divide the lower bound for the number of distinct Room squares of side s obtained in Theorem 3.3 by the number of possible distinct Room squares isomorphic to a given Room square of side s . This number can be at most $((9n+2t)!)^2 (9n+2t+1)!$ (allowing permutations of rows, columns and symbols). Thus,

$$\begin{aligned} \text{NR}(s) &= \text{NR}(9n+2t) \geq \frac{(9!)^{(n-t)n} (8! (3!)^3)^{tn} (2(n!)^2)^8 2((n+2t)!)^2}{((9n+2t)!)^2 (9n+2t+1)!} \\ &> \frac{(9!)^{n^2} 2^9 (n!)^{18}}{((11n)!)^2 (11n+1)!}. \end{aligned}$$

We examine the behavior of the above quantity with the following version of Stirling's formula (see [9]).

LEMMA 3.5. $n! = (2\pi n)^{1/2} (n/e)^n e^{\alpha_n}$ where $1/(12n+1) < \alpha_n < 1/12n$.

THEOREM 3.6. Suppose there is a $\text{TD}(9, n)$ and $9n \leq s \leq 11n$ where n and s are odd. Then

$$\text{NR}(s) > c \cdot \exp\left(n^2 \ln(9!) - 15n \ln(n) - (33 \ln 11 - 15)n + \frac{13}{2} \ln(n)\right),$$

where $c = (2^{29} \pi^{15} / (9 \cdot 11^3))^{1/2}$.

PROOF. From Corollary 3.4, we have

$$\begin{aligned} \text{NR}(s) &> \frac{2^9 \cdot (9!)n^2 \cdot (n!)^{18}}{((11n)!)^3 (11n+1)} \\ &\geq \frac{2^9 (9!)^{n^2} (2\pi n)^9 (n/e)^{18n} e^{18/(12n+1)}}{(22\pi n)^{3/2} (11n/e)^{33n} e^{3/(12 \cdot 11n)} (11n+1)} \quad (\text{Lemma 3.5}) \\ &> \frac{2^9 (9!)^{n^2} (2\pi n)^9 (n/e)^{18n}}{(22\pi n)^{3/2} (11n/e)^{33n} \cdot 12n} \\ &= \left(\frac{2^{29} \cdot \pi^{15}}{9 \cdot 11^3} \right)^{1/2} \exp\left(n^2 \ln(9!) - 15n \ln(n) - (33 \ln 11 - 15)n + \frac{13}{2} \ln(n)\right). \end{aligned}$$

The above bound is dependent upon the existence of certain transversal designs. The following result is well known.

LEMMA 3.7. Let n have prime power factorization $\pi p_i^{\alpha_i}$. If $k \leq 1 + \min\{p_i^{\alpha_i}\}$, then there exists a $\text{TD}(k, n)$.

One can then prove the following by use of the above lemma and a simple arithmetic argument (see [13]).

COROLLARY 3.8. If $s \geq 153$, then there is a positive integer n , with $9n \leq s \leq 11n$, such that a $\text{TD}(9, n)$ exists.

Elementary calculus shows that the bound of Corollary 3.6 is an increasing function of n (for $n \geq 9$, for example). Thus we have

THEOREM 3.9. For $s \geq 153$ odd,

$$\text{NR}(s) > c \cdot \exp\left(\frac{s^2}{121} \ln(9!) - \frac{15}{11} s \ln(s) + \left(\frac{15}{11} - \frac{18}{11} \ln(11)\right)s + \frac{13}{2} \ln(s)\right),$$

where $c = (2^{29} \pi^{15} / (9 \cdot 11^{16}))^{1/2} > .19$.

COROLLARY 3.10. (1) For $s \geq 153$, s odd, $\text{NR}(s) > .19e^{.04s^2} > 1$.

(2) For $s \geq 1001$, s odd, $\text{NR}(s) > .19e^{.09s^2}$.

(3) $\text{NR}(1001) > 10^{40850}$.

4. Summary and remarks. We have shown that the number of nonisomorphic (and inequivalent) Room squares grows extremely rapidly. The techniques in this paper are quite general. A similar argument involving group-divisible designs could be used, for example, to show that the number of nonisomorphic $(v, k, 1)$ balanced incomplete block designs approaches infinity as v approaches infinity.

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