ONE EXAMPLE OF THE BOUNDARY BEHAVIOUR OF BIHOLOMORPHIC TRANSFORMATIONS

B. L. FRIDMAN

ABSTRACT. Two biholomorphically equivalent domains $\Omega_1, \Omega_2 \subset \mathbb{C}^2$ with piecewise smooth boundaries and with the following property are constructed. If $F: \Omega_1 \rightarrow \Omega_2$ is any biholomorphic transformation then neither F nor F^{-1} can be extended continuously to the boundary.

0. In the theory of functions of one complex variable the following statement (Carathéodory theorem) is true. Let $f: \Omega_1 \rightarrow \Omega_2$ be a biholomorphic mapping, $\Omega_1, \Omega_2 \subset \mathbb{C}^1$ are domains bounded by piecewise smooth simple closed curves. Then f admits a continuous extension up to the boundary $\partial\Omega_1$ of Ω_1 . In $\mathbb{C}^n, n > 1$, for some types of domains (say, some types of pseudoconvex domains with smooth boundaries or analytic polyhedra, see [1–6]) an analogous theorem is true. But actually in $\mathbb{C}^n, n > 1$, the situation is different (for further discussions see [7]).

The purpose of this paper is to construct two bounded domains $\Omega_1, \Omega_2 \subset \mathbb{C}^2$ with the following properties.

(P₁) Ω_1 and Ω_2 have piecewise smooth boundaries homeomorphic to the sphere $\partial B^2 = \{z \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$.

(P₂) Both Ω_1 and Ω_2 are biholomorphically equivalent to the polydisc $D^2 = \{z \in \mathbb{C}^2 \mid |z_j| < 1, j = 1, 2\}$.

(P₃) There exists a C^∞ one-to-one mapping $f: \bar{\Omega}_1 \rightarrow \bar{\Omega}_2$ ($\bar{\Omega}$ is the closure of Ω).

(P₄) For any biholomorphic mapping $F: \Omega_1 \rightarrow \Omega_2$ neither F nor F^{-1} can be extended continuously to the boundary.

REMARK. The family Φ of such biholomorphic mappings is quite large. If $G_k: D^2 \rightarrow \Omega_k, k = 1, 2$, are fixed biholomorphic mappings, then $\Phi = \{G_2 \circ G \circ G_1^{-1} \mid G \in \text{Aut } D^2\}$, where $\text{Aut } D^2$ is the group of holomorphic automorphisms of D^2 . So, Φ is as “large” as $\text{Aut } D^2$ which is a 6-parameter group (for description of $\text{Aut } D^2$ see [7]).

1. Definition of $\Omega_k, k = 1, 2$.

(A) First we choose two sets of points $\zeta_{sk} \in \mathbb{C}, s = 1, 2, 3, 4, k = 1, 2$, with the following properties:

(A₁) $|\zeta_{sk}| = 1$ for all s, k .

(A₂) There does not exist a biholomorphic automorphism g of the unit disc, $g: D \rightarrow D$, such that $g(\bigcup_{s=1}^4 \zeta_{s1}) = \bigcup_{s=1}^4 \zeta_{s2}$.

Received by the editors January 21, 1983.

1980 *Mathematics Subject Classification*. Primary 32H99.

©1983 American Mathematical Society
0002-9939/83 \$1.00 + \$.25 per page

REMARK. Every $g \in \text{Aut } D$ can be extended analytically to the boundary ∂D . So, in fact, in (A_2) $g: \bar{D} \rightarrow \bar{D}$.

It is easy to see that such points $\{\zeta_{sk}\}$ exist.

(B) Consider now two mappings $F_k: D^2 \rightarrow \mathbb{C}^2$;

$$(1) \quad F_k: \begin{cases} w_1 = z_1, \\ w_2 = z_2 \prod_{s=1}^4 (\zeta_{sk} - z_1), \end{cases} \quad k = 1, 2.$$

We set now $\Omega_k = F_k(D^2)$, $k = 1, 2$.

2. THEOREM. Ω_k , $k = 1, 2$, have properties (P_1) – (P_4) .

PROOF. (a) (P_1) and (P_2) are almost obvious and follow from the description of F_k .

(b) To prove (P_3) let us take a C^∞ mapping $g: \bar{D} \rightarrow \bar{D}$ which is one-to-one and such that $g(\cup_{k=1}^4 \zeta_{k1}) = \cup_{k=1}^4 \zeta_{k2}$. We set $G: (z_1, z_2) \rightarrow (g(z_1), z_2)$. Then the mapping $F_2 \circ G \circ F_1^{-1}: \Omega_1 \rightarrow \Omega_2$ can be extended to a C^∞ one-to-one mapping $\bar{\Omega}_1 \rightarrow \bar{\Omega}_2$.

(c) Now we check property (P_4) . Let $F: \Omega_1 \rightarrow \Omega_2$ be a biholomorphic mapping. Let us consider $G = F_2^{-1} \circ F \circ F_1: D^2 \rightarrow D^2$. Evidently $G \in \text{Aut } D^2$. From the description of this group $G = (g_1, g_2)$, where $g_k \in \text{Aut } D$, $k = 1, 2$, and either $g_1 = g_1(z_1)$, $g_2 = g_2(z_2)$ or $g_1 = g_1(z_2)$, $g_2 = g_2(z_1)$. Let $g = g(z_1)$ denote g_1 if we are dealing with the first case, and g_2 otherwise. Since property (A_2) holds for $\{\zeta_{sk}\}$ we can always find an integer j , $1 \leq j \leq 4$, such that

$$(2) \quad g(\zeta_{j1}) \neq \zeta_{s2} \quad \text{for any } s, 1 \leq s \leq 4.$$

Let us now take a sequence $\{\lambda_n\}_{n=1}^\infty \subset D$ such that $\lim_{n \rightarrow \infty} \lambda_n = \zeta_{j1}$, and consider two sequences $u_n = F_1(\lambda_n, \frac{1}{2})$ and $v_n = F_1(\lambda_n, -\frac{1}{2})$, $1 \leq n < \infty$. Both of them lie in Ω_1 and, according to (1),

$$(3) \quad \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n = (\zeta_{j1}, 0) \in \partial\Omega_1.$$

On the other hand, in the first case ($g = g_1$) we have

$$\lim_{n \rightarrow \infty} F(u_n) = \lim_{n \rightarrow \infty} (F_2 \circ G \circ F_1^{-1})(u_n) = F_2\left(g(\zeta_{j1}), g_2\left(\frac{1}{2}\right)\right)$$

and

$$\lim_{n \rightarrow \infty} F(u_n) = F_2\left(g(\zeta_{j1}), g_2\left(-\frac{1}{2}\right)\right).$$

From (2) and the description of F_2 ,

$$(4) \quad \lim_{n \rightarrow \infty} F(u_n) \neq \lim_{n \rightarrow \infty} F(v_n).$$

The second case ($g = g_2$) can be considered similarly. In both cases (4) holds. (P_4) now follows from (3), (4) and similar considerations for F^{-1} .

REFERENCES

1. S. Bell, *Smooth bounded strictly and weakly pseudoconvex domains cannot be biholomorphic*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), 119–120.
2. S. Bell and D. Cathlin, *Proper holomorphic mappings extend smoothly to the boundary*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 269–272.

3. K. Diederich and J. E. Fornaess, *Smooth extendability of proper holomorphic mappings*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 264–268.
4. C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65.
5. B. L. Fridman, *On a class of analytic polyhedra*, Dokl. Akad. Nauk SSSR **242** (1978); English transl., Soviet Math. Dokl. **19** (1979), 1258–1261.
6. G. Henkin, *An analytic polyhedron is not holomorphically equivalent to a strictly pseudoconvex domain*, Soviet. Math. Dokl. **14** (1973), 858–862.
7. S. G. Krantz, *Function theory of several complex variables*, Wiley, New York, 1982.

DEPARTMENT OF MATHEMATICS, WICHITA STATE UNIVERSITY, WICHITA, KANSAS 67208