CROSSED PRODUCTS OF C^* -DYNAMICAL SYSTEMS WITH GROUND STATES

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ABSTRACT. Let $C^*(A, \mathbf{R})$ be the crossed product associated with a C^* -dynamical system $(A, \mathbf{R}, \alpha)$. We show that if there is a ground state for A , then there is an ideal I in $C^*(A, \mathbf{R})$ such that if $\lambda_1 < \lambda_2$, then $\hat{\alpha}_{\lambda_1}(I) \subsetneq \hat{\alpha}_{\lambda_2}(I)$, where $\hat{\alpha}$ is the dual automorphism group on $C^*(A, \mathbf{R})$, and such that $C^*(A, \mathbf{R}) = \overline{\bigcup_{\lambda \geq 0} \hat{\alpha}_{\lambda}(I)}$. Conversely, we show that if A is unital, and if $C^*(A, \mathbf{R})$ has such a decomposition for some proper ideal I , then A has a ground state.

1. Introduction. Let A be a C^* -algebra, let G be a locally compact group, and let $\{\alpha_t\}_{t \in G}$ be an action of G on A by $*$ -automorphisms; i.e., assume that $t \rightarrow \alpha_t$ is a homomorphism of G into $\text{Aut}(A)$ such that $t \rightarrow \alpha_t(x)$ is continuous for all $x \in A$. We refer to (A, G, α) as a C^* -dynamical system and we denote the associated crossed product C^* -algebra by $C^*(A, G)$ (see § 2 or [5]). Recently, the ideal structure of $C^*(A, G)$ has attracted considerable attention, particularly in those cases when G is abelian; for when G is abelian, spectral theory may be applied to good effect (cf. [5 or 6]). In the cases when $G = \mathbf{R}$, A is UHF, and $\{\alpha_t\}_{t \in \mathbf{R}}$ is of product type, Bratteli [1] proved that $C^*(A, \mathbf{R})$ is not simple since A admits a ground state for α ; i.e., a state ϕ such that for all $x \in A$ and $y \in A^a$, the space of entire analytic elements in A , the function $f(t) = \phi(x\alpha_t(y))$ extends to be analytic in the upper half-plane and bounded there by $\|x\|\|y\|$. Subsequently, Pedersen and Takai [7] extended this result to the case when A and α are arbitrary (but $G = \mathbf{R}$). Our primary objective in this note is to prove the results described in the abstract which generalize the Pedersen-Takai's result and provide a converse. In the last section, we shall generalize our results to cover certain C^* -dynamical system (A, G, α) where $\hat{G}$ is totally ordered.

2. Preliminaries. Let (A, G, α) be a C^* -dynamical system. The C^* -crossed product $C^*(A, G)$ for (A, G, α) is defined as the enveloping C^* -algebra of $L^1(A, G)$, the set of all Bochner integrable A -valued functions on G equipped with the following Banach $*$ -algebra structure:

$$(xy)(t) = \int_G x(s)\alpha_s(y(s^{-1}t)) \, ds,$$

$$x^*(t) = \Delta(t)^{-1}\alpha_t(x(t^{-1}))^*,$$

$$\|x\|_1 = \int_G \|x(s)\| \, ds,$$

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where ds is the left Haar measure of G and $\Delta(t)$ is the associated modular function. From now on, we assume that G is a locally compact abelian group. The dual action $\hat{\alpha}$ of α is defined on $C^*(A, G)$ by the formula

$$\hat{\alpha}_\gamma(x)(t) = \overline{\langle \gamma, t \rangle} x(t),$$

where $\gamma \in \hat{G}$, $x \in L^1(A, G)$, and where $\langle \gamma, t \rangle$ is the value of γ at t . Now let ϕ be an α -invariant state over (A, G, α) . We denote the cyclic covariant representation associated with ϕ by $(\pi_\phi, u^\phi, H_\phi, \xi_\phi)$, i.e., π_ϕ is the GNS representation of A on the Hilbert space H_ϕ , with the canonical cyclic vector ξ_ϕ , and u^ϕ is defined by

$$u_t^\phi \pi_\phi(x) \xi_\phi = \pi_\phi(\alpha_t(x)) \xi_\phi$$

for $x \in A$, $t \in G$ (see [5, 7.4.12]). Then there is a unique spectral measure μ on the Borel sets of the dual group $\hat{G}$ with values in $B(H_\phi)$ such that

$$u_t^\phi = \int_{\hat{G}} \langle \gamma, t \rangle d\mu(\gamma),$$

where t is an element of G (see [5, 8.3.2]). Here, we remark that $\text{Sp}(u^\phi) = \text{supp of } \mu$, where Sp denotes spectrum in the sense of spectral synthesis. The corresponding representation $(\pi_\phi \times u^\phi, H_\phi, \xi_\phi)$ of $C^*(A, G)$ is defined by

$$(\pi_\phi \times u^\phi)(x) = \int_G \pi_\phi(x(t)) u_t^\phi dt$$

for any $x \in L^1(A, G)$ (see [5, 7.6.4]).

3. Results.

PROPOSITION 3.1. *Let $(A, \mathbf{R}, \alpha)$ be a C^* -dynamical system and let $\hat{\alpha}$ denote the dual action of $\mathbf{R}$ on $C^*(A, \mathbf{R})$. If A has a ground state, then there is a proper, closed, two-sided ideal I in $C^*(A, \mathbf{R})$ such that if $\lambda_1 < \lambda_2$, then $\hat{\alpha}_{\lambda_1}(I) \subsetneq \hat{\alpha}_{\lambda_2}(I)$ and for any $\lambda > 0$, $C^*(A, \mathbf{R}) = \overline{\bigcup_{n \in \mathbf{Z}_+} \hat{\alpha}_{n\lambda}(I)}$, where $\mathbf{Z}_+$ denotes the nonnegative integers.*

PROOF. We use the notation in §2. Let ϕ be a ground state. From §2, u_t^ϕ has the spectral decomposition

$$u_t^\phi = \int_{\mathbf{R}} e^{its} dE(s),$$

and the support of $E = \text{Sp}(u^\phi)$ is contained in $\mathbf{R}_+$, where $\mathbf{R}_+$ is the set of all nonnegative real numbers (see [5, 8.12.5]). For each $k \geq 0$, $\mathbf{R} \ni t \rightarrow e^{ikt} u_t^\phi$ is a unitary representation on H_ϕ . Put $v_k(t) = e^{ikt} u_t^\phi$. Then, (π_ϕ, v_k, H_ϕ) is a covariant representation. Hence, we can consider the corresponding representation $(\pi_\phi \times v_k, H_\phi)$. We denote the direct sum of $\{\pi_\phi \times v_k\}_{k \geq 0}$ by $\bigoplus_{k \geq 0} (\pi_\phi \times v_k)$. Let

$$I = \ker \left(\bigoplus_{k \geq 0} (\pi_\phi \times v_k) \right)$$

be the kernel of $\bigoplus_{k \geq 0} (\pi_\phi \times v_k)$. Now fix a positive number λ arbitrarily. Then we have

$$\begin{aligned}\hat{\alpha}_\lambda(I) &= \hat{\alpha}_\lambda \left(\ker \left(\bigoplus_{k \geq 0} (\pi_\phi \times v_k) \right) \right) \\ &= \ker \left(\left(\bigoplus_{k \geq 0} (\pi_\phi \times v_k) \right) \circ \hat{\alpha}_{-\lambda} \right) = \ker \left(\bigoplus_{k \geq 0} (\pi_\phi \times v_{k+\lambda}) \right) \\ &= \ker \left(\bigoplus_{k \geq \lambda} (\pi_\phi \times v_k) \right) \supset \ker \left(\bigoplus_{k \geq 0} (\pi_\phi \times v_k) \right) = I.\end{aligned}$$

Take any f in $L^1(\mathbf{R})$. By an easy consequence of Plancherel's theorem, there is a sequence $\{f_n\}$ in $L^1(\mathbf{R})$ such that $\text{supp}(\hat{f}_n)$ is compact for each n and $\|f_n - f\|_{L^1} \rightarrow 0$, where $\hat{f}_n$ denotes the inverse Fourier transform of f_n . Since $\text{supp}(\hat{f}_n)$ is compact, we can choose a natural number $m(n)$ such that $\text{supp}(\hat{f}_n(\cdot + m(n)\lambda))$ is contained in $\mathbf{R}_-$ for each f_n , where $\mathbf{R}_-$ is the set of all negative numbers. In particular, $\text{supp}(\hat{f}_n(\cdot + m(n)\lambda + k))$ is contained in $\mathbf{R}_-$ for any $k \geq 0$. Put $g_n(t) = e^{im(n)\lambda t} f_n(t)$ and $(x \otimes g)(t) = g(t)x$ for any g in $L^1(\mathbf{R})$ and any x in A . Then we have

$$(*) \quad \hat{\alpha}_{m(n)\lambda}(x \otimes g_n)(t) = (x \otimes f_n)(t).$$

For any $k \geq 0$,

$$\begin{aligned}(\pi_\phi \times v_k)(x \otimes g_n) &= \int_{\mathbf{R}} \pi_\phi(g_n(t)x) v_k(t) dt \\ &= \pi_\phi(x) \int_{\mathbf{R}} g_n(t) e^{ikt} u_t^\phi dt \\ &= \pi_\phi(x) \int \int g_n(t) e^{ikt} e^{its} dt dE(s) \\ &= \pi_\phi(x) \int_{\mathbf{R}} \hat{f}_n(s + m(n)\lambda + k) dE(s) = 0,\end{aligned}$$

since the support of E is contained in $\mathbf{R}_+$. Therefore,

$$\left(\bigoplus_{k \geq 0} (\pi_\phi \times v_k) \right) (x \otimes g_n) = 0$$

for any n . Since $x \otimes g_n$ belongs to I , we have

$$\{x \otimes f_n\} \subset \bigcup_{\mathbf{Z}_+ \ni m} \hat{\alpha}_{m\lambda}(I)$$

from (*). Since

$$\begin{aligned}\|x \otimes f_n - x \otimes f\| &\leq \|x \otimes f_n - x \otimes f\|_{L^1} \\ &= \|x\| \|f_n - f\|_{L^1} \rightarrow 0,\end{aligned}$$

we have $x \otimes f \in \overline{\bigcup_{\mathbf{Z}_+ \ni n} \hat{\alpha}_{n\lambda}(I)}$. Now we easily see that the ideal I is strictly contained in $\hat{\alpha}_\lambda(I)$. Thus, if $\lambda_1 < \lambda_2$, then we have $\hat{\alpha}_{\lambda_1}(I) \subsetneq \hat{\alpha}_{\lambda_2}(I)$ and $L^1(A, \mathbf{R}) \subset \overline{\bigcup_{\mathbf{Z}_+ \ni n} \hat{\alpha}_{n\lambda}(I)}$. Therefore, we see that $C^*(A, \mathbf{R}) = \overline{\bigcup_{\mathbf{Z}_+ \ni n} \hat{\alpha}_{n\lambda}(I)}$. Q.E.D.

REMARK 3.2. As seen from the above proof, the ideal I satisfies $I \subset \hat{\alpha}_\lambda(I)$ for any $\lambda \geq 0$. Of course, we have $C^*(A, \mathbf{R}) = \overline{\bigcup_{\mathbf{R} \ni \lambda} \hat{\alpha}_\lambda(I)}$.

We recall that for any proper ideal, there is a primitive ideal containing it. Hence, we obtain the following.

COROLLARY 3.3. *Let $(A, \mathbf{R}, \alpha)$ be a C^* -dynamical system. If A has a ground state, then $C^*(A, \mathbf{R})$ contains a primitive ideal ζ such that for any $\lambda > 0$, $C^*(A, \mathbf{R}) = \bigcup_{n \in \mathbf{Z}_+} \hat{\alpha}_{n\lambda}(\zeta)$.*

Assume that a C^* -algebra A is unital. Then, Powers and Sakai [8, 2.3] proved that there exists a ground state for α if α is approximately inner.

COROLLARY 3.4. *Let $(A, \mathbf{R}, \alpha)$ be a C^* -dynamical system, where $A \ni 1$. If α is approximately inner, the statement of Corollary 3.3 holds.*

The above corollary is an extension of the result of Bratteli [1, 3.2], in which he assumed that A is a UHF algebra with product type action of $\mathbf{R}$. Moreover, under the same assumptions, he showed the same result for an abelian, connected, compact group such that the cardinality of the dual group does not exceed the power of the continuum, instead of $\mathbf{R}$. As for this case, we shall state a remark in §4.

Here, we consider the converse of Remark 3.2 and show how the ideal structure of $C^*(A, \mathbf{R})$ is related to the presence of ground states on A .

THEOREM 3.5. *Let $(A, \mathbf{R}, \alpha)$ be a C^* -dynamical system, where A is a unital C^* -algebra. Then the following conditions are equivalent:*

- (i) *There is an α -ground state of A .*
- (ii) *There is a proper, closed, two-sided ideal I of $C^*(A, \mathbf{R})$ such that $I \subset \hat{\alpha}_\lambda(I)$ for any $\lambda \geq 0$ and the union of $\hat{\alpha}_\lambda(I)$ with all $\lambda \in \mathbf{R}$ is dense in $C^*(A, \mathbf{R})$.*

PROOF. (i) $\Rightarrow$ (ii). This follows from Remark 3.2.

(ii) $\Rightarrow$ (i). Let B be an α -invariant hereditary C^* -subalgebra of A . Put $J_B = \{f \in L^1(\mathbf{R}) \mid x \otimes f \in I \text{ for any } x \in B\}$. Then, J_B is a closed ideal. Set $Z(J_B) = \{t \in \mathbf{R} \mid \hat{f}(t) = 0 \text{ for any } f \in J_B\}$. Since $I \subset \hat{\alpha}_\lambda(I)$ for any $\lambda \geq 0$, we have $J_B \subset \hat{\alpha}_\lambda(J_B)$. Hence, when $Z(J_B)$ is not empty, we have $Z(J_B) \supset Z(J_B) + \lambda$ for any $\lambda \geq 0$. Thus, if $Z(J_B)$ is neither empty nor $\mathbf{R}$, it is a half-line (i.e., it is of the form $[r, +\infty)$, where r is a real number). We denote the greatest lower bound (including $+\infty$) of $Z(J_B)$ by $m(B)$, where we set $m(B) = +\infty$ (resp. $-\infty$) in the case that $Z(J_B)$ is empty (resp. $\mathbf{R}$). Let n be a positive integer and let H_n be the set of α -invariant hereditary C^* -subalgebras B of A with $m(B) \geq -n$. When B_n denotes the α -invariant hereditary C^* -subalgebra generated by those in H_n , we have $B_n \in H_n$. Let $\tilde{B}_n$ be the linear span of elements of the form axb with $a \in B' \in H_n$, $b \in B'' \in H_n$, $x \in A$. Since $\tilde{B}_n$ is dense in B_n , it follows that $y \otimes g \in I$ for all $y \in B_n$ and $g \in L^1(\mathbf{R})$ with $\text{supp}(\hat{g}) \subset (-\infty, -n)$.

We assert that $\bigcup_n B_n$ is dense in A . Otherwise, denoting the closure of $\bigcup_n B_n$ by B_∞ , we have an α -invariant state ϕ of A such that $\phi = 0$ on B_∞ since B_∞ is α -invariant. Through the representation $\pi_\phi \times u^\phi$ of $C^*(A, \mathbf{R})$, define a state ϕ of

$C^*(A, \mathbf{R})$ by the cyclic vector ξ_ϕ , where $(\pi_\phi, u^\phi, H_\phi, \xi_\phi)$ is the cyclic covariant representation of A via ϕ . By Kishimoto's result (see [3, Theorem 1]) the ideal I is densely spanned by elements of the form $x \otimes f$ with $x \in A$, $f \in L^1(\mathbf{R})$. Therefore, there are $x \otimes f$ in I and $\lambda \geq 0$ such that

$$\bar{\phi}(\hat{\alpha}_\lambda(x \otimes f)) \neq 0 \quad (\text{i.e., } \phi(x) \neq 0).$$

Let $H(x^*Ax)$ be the α -invariant hereditary C^* -subalgebra generated by x^*Ax . Then, $H(x^*Ax) \in H_n$ for some $n > 0$, which implies $\phi(x^*x) = 0$ (i.e., $\phi(x) = 0$). Thus, we have reached a contradiction. Therefore, $\bigcup_n B_n$ is dense in A .

Suppose $B_n \neq A$ for any n . Since B_n is the hereditary C^* -subalgebra of A for each n , $\|x - 1\| \geq 1$ for any $x \in B_n$, which implies $1 \notin B_\infty = \overline{\bigcup_n B_n}$. Hence, $B_n = A$ for some $n > 0$. Therefore, if $\text{supp}(\hat{f}) \subset (-\infty, -n)$, we have $1 \otimes f \in I$.

Now we choose a covariant representation (π, u, H) of A such that $\ker(\pi \times u) \supset I$. Then, $\text{Sp}(u) \subset [-n, +\infty)$. Put

$$(*) \quad \delta = \inf(\text{Sp}(u)).$$

We take a sequence $\{\xi_k\}$ in H such that $\text{Sp}_u(\xi_k) \subset [\delta, \delta + k^{-1}]$ and $\|\xi_k\| = 1$ for each k . Putting $\phi_k(x) = (\pi(x)\xi_k | \xi_k)$, then we have a weak* limit point ϕ of $\{\phi_k\}$ in the state space of A . We take an arbitrary positive number ε . If $x \in A$, with $\text{Sp}_\alpha(x) \subset (-\infty, -\varepsilon)$, then

$$\text{Sp}_u(\pi(x)\xi_k) \subset \text{Sp}_\alpha(x) + \text{Sp}_u(\xi_k)$$

(see [5, 8.2.4]), and so it follows easily from (*) that $\phi_k(x^*x) = 0$ for $k^{-1} < \varepsilon$. (See [2 or 5] for the spectral theory.) Hence, we have $\phi(x^*x) = 0$ for all $x \in A$ with $\text{Sp}_\alpha(x) \subset (-\infty, 0)$. Q.E.D.

4. Appendix. Let G be a locally compact abelian group. Suppose P is a semigroup in the dual group $\hat{G}$ which is closed and has two additional properties;

$$P \cap (-P) = \{0\}, \quad P \cup (-P) = \hat{G}$$

(P is said to be a positive cone). Under these conditions, P induces an order in $\hat{G}$, i.e., $\gamma \geq \gamma'$ if $\gamma - \gamma' \in P$ (see [9]).

Let (A, G, α) be a C^* -dynamical system. Then if a state ϕ is α -invariant and $\text{Sp}(u^\phi) \subset P$, we call ϕ a ground state for (G, P) .

Now let G be an abelian, connected, compact group. Assume that the cardinality of $\hat{G}$ does not exceed the power of the continuum. Then, $\hat{G}$ has a positive cone P which induces an archimedean order in $\hat{G}$ (see [9, 8.1.2]). Now we have the following result:

If A has a ground state for (G, P) where P induces an archimedean order in $\hat{G}$, then $C^*(A, G)$ contains a proper, closed, two-sided ideal I such that $\hat{\alpha}_{\lambda_1}(I) \subsetneq \hat{\alpha}_{\lambda_2}(I)$ for $\lambda_1 < \lambda_2$ in $\hat{G}$ and if $\lambda > 0$, then

$$C^*(A, G) = \overline{\bigcup_{n \in \mathbf{Z}_+} \hat{\alpha}_{n\lambda}(I)}.$$

Hence, the corresponding result for Corollary 3.3 also holds.

We sketch the proof. By [9, 8.1.2], there is an order preserving isomorphism of $\hat{G}$ onto a subgroup of $\mathbf{R}$. Hence, it is easily seen that there exists τ in P such that $\bigcap_n (n\tau + P)$ is empty, i.e., $\bigcup_n (n\tau + P^c) = \hat{G}$, where P^c is the complement of P in $\hat{G}$. If $\gamma \geq \gamma' \geq 0$, $\gamma + P^c \supset \gamma' + P^c$. Hence, $\{n\tau + P^c\}$ is an increasing sequence. For any f in $L^1(G)$, there exists a sequence $\{f_n\}$ in $L^1(G)$ such that $\text{supp}(\hat{f}_n)$ is compact for each n and $\|f_n - f\|_{L^1} \rightarrow 0$, where $\hat{f}_n$ is the inverse Fourier transform of f_n (see [9]). Then, for each n , there exists a natural number $k(n)$ such that $\text{supp}(\hat{f}_n) \subset k(n)\tau + P^c$. Hence, we have $\text{supp}(\hat{f}_n(\cdot + k(n)\tau + \gamma)) \subset P^c$ for any γ in P . Thus, we can obtain the desired result by the same method used in §3.

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