SETS OF UNIQUENESS FOR A CERTAIN CLASS $\mathfrak{N}_\epsilon$ ON THE DYADIC GROUP

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ABSTRACT. For each sequence $\epsilon = \{\epsilon_n\}$ of real numbers which satisfies $\liminf_{n \rightarrow \infty} \epsilon_{2^{n+1}}/\epsilon_{2^n} > 0$ and $\epsilon_n \downarrow 0$ as $n \rightarrow \infty$, let $\mathfrak{N}_\epsilon$ denote the set of all Walsh series $\mu \sim \sum_{k=0}^{\infty} \hat{\mu}(k)w_k(x)$ such that $\sum_{k=0}^{\infty} \epsilon_k |\hat{\mu}(k)|^2 < \infty$. We give a necessary and sufficient condition for a subset of the dyadic group to be a set of uniqueness for $\mathfrak{N}_\epsilon$.

1. Introduction. Wade [3] discussed whether or not a subset of the dyadic group is a set of uniqueness for $\mathfrak{T}_\alpha^+$ and gave the complete solution for this problem. In this paper we shall partially generalize his result.

First we introduce some definitions and notations. Let I_0^0 be the dyadic group introduced by Fine [1], i.e., a compact abelian group with the additive operation $+$. A dyadic interval of rank n , I_n^p , is the set of all 0-1 series, $x = (t_1, t_2, \dots)$, such that $\sum_{k=1}^n t_k/2^k = p/2^n$. The dyadic interval of rank n which contains x is denoted by $I_n(x)$. For convenience, when $\lim_{k \rightarrow \infty} t_k \neq 1$, we shall identify $x = (t_1, t_2, \dots)$ with $(\sum_{k=1}^{\infty} t_k/2^k)$, and when $\lim_{k \rightarrow \infty} t_k = 1$, we shall write $x = (\sum_{k=1}^{\infty} t_k/2^k)^-$.

Let

$$\mu \sim \sum_{k=0}^{\infty} \hat{\mu}(k)w_k(x)$$

be an arbitrary Walsh series. Set

$$m_\mu(I) = \lim_{N \rightarrow \infty} \int_I \sum_{k=0}^N \hat{\mu}(k)w_k(x) dx$$

for each dyadic interval I . The limit of the right-hand side exists and is finite. Moreover, we have

$$m_\mu(I_n(x)) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x).$$

The set function m_μ satisfies the following additive law:

$$(1) \quad m_\mu(I_n^p) = m_\mu(I_{n+1}^{2p}) + m_\mu(I_{n+1}^{2p+1})$$

for $n = 0, 1, 2, \dots$, $p = 0, 1, \dots, 2^n - 1$. By an easy calculation we can prove that

$$\hat{\mu}(k) = \int_{I_0^0} w_k(x) m_\mu(dx),$$

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where $\int_I m_\mu(dx) = m_\mu(I)$. Conversely, when a set function m satisfies (1), if we set

$$\hat{\gamma}(k) = \int_{I_0^0} w_k(x) m(dx)$$

for all k , then

$$\gamma \sim \sum_{k=0}^{\infty} \hat{\gamma}(k) w_k(x)$$

is the Walsh Fourier series of $m = m_\gamma$. We call a set function satisfying (1) a *dyadic measure*. For details of dyadic measures and the dyadic group, see [1 and 4].

When $\varepsilon \equiv \{\varepsilon_k\}_k$ is a sequence of positive numbers such that

$$\begin{cases} \varepsilon_0 = 1, & \varepsilon_k \downarrow 0 \text{ as } k \rightarrow \infty, \\ \liminf_{n \rightarrow \infty} \varepsilon_{2^{n+1}} / \varepsilon_{2^n} \equiv \theta > 0, \end{cases}$$

let $\mathfrak{M}_\varepsilon$ denote the set of all Walsh series μ which satisfy $\sum_{k=0}^{\infty} \varepsilon_k |\hat{\mu}(k)|^2 < \infty$. A subset E of the dyadic group is said to be a *set of uniqueness* for $\mathfrak{M}_\varepsilon$ if $\mu \in \mathfrak{M}_\varepsilon$ and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0, \quad \text{except perhaps on } E,$$

imply μ is the zero series. If a subset is not a set of uniqueness for $\mathfrak{M}_\varepsilon$, it is called a *set of multiplicity* for $\mathfrak{M}_\varepsilon$. Notice when $\varepsilon_0 = 1$, $\varepsilon_k = 1/k^{1-\alpha}$, $k = 1, 2, \dots$, where $0 < \alpha < 1$, that $\mathfrak{M}_\varepsilon$ coincides with $\mathfrak{T}_\alpha$ (see [3]). Since $\mathfrak{T}_\alpha^+ \subset \mathfrak{T}_\alpha$, if a subset of measure zero is a set of multiplicity for $\mathfrak{T}_\alpha^+$, then it is also a set of multiplicity for $\mathfrak{T}_\alpha$. Since the characteristic function f of a set of positive measure is an L^2 -integrable function, it satisfies

$$\sum_{k=0}^{\infty} \varepsilon_k |\hat{f}(k)|^2 \leq \sum_{k=0}^{\infty} |\hat{f}(k)|^2 < \infty.$$

Hence any set of positive measure is a set of multiplicity for each $\mathfrak{M}_\varepsilon$.

We shall prove the following theorems.

THEOREM 1. *Let $\{n_k\}_k$ be a sequence of integers and set $N_k = n_1 + n_2 + \dots + n_k$, $N_0 = 0$. Suppose E is a subset of I_0^0 which satisfies the following property.*

$$\begin{aligned} E \cap I_{N_k+1}^{2^{p+s(p,k)}} &= \emptyset \quad \text{for } k = 1, 2, \dots, p = 0, 1, 2, \dots, 2^{N_k} - 1 \\ &\text{and } s(p, k) = 0 \text{ or } 1. \end{aligned}$$

Then E is a set of uniqueness for $\mathfrak{M}_\varepsilon$ if and only if $\sum_{k=1}^{\infty} \varepsilon_{2^{N_k}} \cdot 2^k = \infty$.

THEOREM 2. *It is sufficient for a closed set of measure zero to be a set of uniqueness for $\mathfrak{M}_\varepsilon$ if it satisfies*

$$\sum_{n=0}^{\infty} \frac{\varepsilon_{2^n} - \varepsilon_{2^{n+1}}}{\text{meas } E_{n+1}} = \infty$$

or

$$\limsup_{n \rightarrow \infty} \varepsilon_{2^n} / \text{meas } E_n = \infty,$$

where $E_n = \bigcup_{x \in E} I_n(x)$.

2. Proof of Theorem 1. Let m_μ be the dyadic measure associated with μ . Since

$$\sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2 = 2^n \sum_{p=0}^{2^n-1} |\Delta m_\mu(I_n^p)|^2,$$

where

$$\Delta m_\mu(I_n^p) = m_\mu(I_{n+1}^{2p}) - m_\mu(I_{n+1}^{2p+1}),$$

we have

$$\sum_{k=2^n}^{2^{n+1}-1} \varepsilon_k |\hat{\mu}(k)|^2 \leq \varepsilon_{2^n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2 = \varepsilon_{2^n} \cdot 2^n \cdot \sum_{p=0}^{2^n-1} |\Delta m_\mu(I_n^p)|^2$$

and

$$\sum_{k=2^n}^{2^{n+1}-1} \varepsilon_k |\hat{\mu}(k)|^2 \geq \varepsilon_{2^{n+1}} \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2 \geq \theta \cdot \varepsilon_{2^n} \cdot 2^n \cdot \sum_{p=0}^{2^n-1} |\Delta m_\mu(I_n^p)|^2.$$

Therefore, it is easy to see that

$$\sum_{k=0}^{\infty} \varepsilon_k |\hat{\mu}(k)|^2 = \infty$$

if and only if

$$\sum_{n=0}^{\infty} \varepsilon_{2^n} \cdot 2^n \sum_{p=0}^{2^n-1} |\Delta m_\mu(I_n^p)|^2 = \infty.$$

Let m be the positive dyadic measure constructed by the following rule when $0 \leq n \leq n_1 = N_1$. Set $m(I_n^p) = 1/2^n$ and

$$m(I_{N_1+1}^p) = \begin{cases} 1/2^{N_1} & \text{if } I_{N_1+1}^p \cap E \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Continuing in this way, when $N_k < n \leq N_{k+1}$, $k = 1, 2, \dots$, set

$$m(I_n^p) = \begin{cases} 1/2^{n-k} & \text{if } I_n^p \cap E \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

From the definitions of m and Δm , we have

$$|\Delta m(I_n^p)| = \begin{cases} 1/2^{N_k-k+1} & \text{if } I_{N_k}^p \cap E \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and when $n \neq N_k$, $\Delta m(I_n^p) = 0$ for all p . Set $m_{\mu_E} = m$. Since $\hat{\mu}_E(0) = 1$, the Walsh series μ_E is not the zero series, and the dyadic measure m_{μ_E} satisfies the following equality:

$$\begin{aligned} \sum_{n=0}^{\infty} \varepsilon_{2^n} \cdot 2^n \sum_{p=0}^{2^n-1} |\Delta m_{\mu_E}(I_n^p)|^2 &= \sum_{k=1}^{\infty} 2^{N_k} \cdot \varepsilon_{2^{N_k}} \cdot 2^{N_k-k+1} (1/2^{N_k-k+1})^2 \\ &= \frac{1}{2} \cdot \sum_{k=1}^{\infty} \varepsilon_{2^{N_k}} \cdot 2^k. \end{aligned}$$

Therefore, if the last formula is finite, E is a set of multiplicity for $\mathfrak{M}_\varepsilon$.

On the other hand, let m' be a dyadic measure satisfying the following properties:

$$(2) \quad (i) m'(I_0^0) = 1; \quad (ii) m'(I) = 0 \quad \text{if } I \cap E = \emptyset,$$

where I is an arbitrary dyadic interval. Therefore we have

$$\begin{aligned} 1 &= |m'(I_0^0)| = \left| \sum_{p=0}^{2^{N_k}-1} m'(I_{N_k}^p) \right| \\ &= \left| \sum_{(I_{N_k}^p \cap E \neq \emptyset)} m'(I_{N_k}^p) \right| \leq \sum_{(I_{N_k}^p \cap E \neq \emptyset)} |\Delta m'(I_{N_k-1}^p)| \\ &\leq \left\{ \sum_{(I_{N_k}^p \cap E \neq \emptyset)} |\Delta m'(I_{N_k-1}^p)|^2 \right\}^{1/2} \cdot \left\{ \sum_{(I_{N_k}^p \cap E \neq \emptyset)} 1 \right\}^{1/2} \\ &= \left\{ \sum_{(I_{N_k}^p \cap E \neq \emptyset)} |\Delta m'(I_{N_k-1}^p)|^2 \right\}^{1/2} \cdot \{2^{(N_k-k)/2}\}. \end{aligned}$$

From the above just cited, we have

$$2^{k-N_k} \leq \sum_{p=0}^{2^{N_k-1}-1} |\Delta m'(I_{N_k-1}^p)|^2.$$

Let μ be the Walsh Fourier series of m' . Then

$$\begin{aligned} \sum_{k=1}^{\infty} \varepsilon_{2^{N_k}} 2^k &\leq \sum_{k=1}^{\infty} \varepsilon_{2^{N_k}} \cdot 2^{N_k} \sum_{p=0}^{2^{N_k}-1} |\Delta m'(I_{N_k-1}^p)|^2 \\ &\leq 2 \cdot \sum_{k=1}^{\infty} \varepsilon_{2^{N_k}} \cdot \sum_{n=2^{N_k-1}}^{2^{N_k}-1} |\hat{\mu}(n)|^2 \\ &\leq 2 \cdot \frac{1}{\theta} \cdot \sum_{k=0}^{\infty} \varepsilon_k \cdot |\mu(k)|^2. \end{aligned}$$

If E satisfies the hypothesis, then m' does not belong to $\mathfrak{N}_\varepsilon$. Namely, E is a set of uniqueness for $\mathfrak{N}_\varepsilon$. The proof is complete.

If the sequence $\{N_k\}_k$ satisfies $\lim_{k \rightarrow \infty} (N_k - k)/N_k = \alpha$, then the Hausdorff dimension of E is α and, for each $\varepsilon > 0$, E satisfies

$$\lim_{k \rightarrow \infty} 2^{N_k-k+1} \cdot (1/2^{N_k+1})^{\alpha+\varepsilon} = 0, \quad \lim_{k \rightarrow \infty} 2^{N_k-k+1} \cdot (1/2^{N_k+1})^{\alpha-\varepsilon} = \infty,$$

that is, the $(\alpha + \varepsilon)$ -Hausdorff measure of E is zero and the $(\alpha - \varepsilon)$ -Hausdorff measure of E is positive. Consequently we know that E is of $(\alpha + \varepsilon)$ -capacity zero and of $(\alpha - \varepsilon)$ -capacity positive (see [2]). A corollary of [3], then, is that E is a set of uniqueness for $\mathfrak{T}_{(\alpha+\varepsilon)}^+$ and a set of multiplicity for $\mathfrak{T}_{(\alpha-\varepsilon)}^+$. Theorem 1 improves this corollary because such a set E satisfies

$$\sum_{k=1}^{\infty} 1/2^{N_k(1-(\alpha+\varepsilon))} \cdot 2^k = \infty, \quad \sum_{k=1}^{\infty} 1/2^{N_k(1-(\alpha-\varepsilon))} \cdot 2^k < \infty.$$

Hence, with $\varepsilon_1 \equiv \{1/k^{(1-(\alpha+\varepsilon))}\}_k$ and $\varepsilon_2 \equiv \{1/k^{(1-(\alpha-\varepsilon))}\}_k$, Theorem 1 implies that E is a set of uniqueness for $\mathfrak{N}_{\varepsilon_1} \equiv \mathfrak{T}_{\alpha+\varepsilon}$ and a set of multiplicity for $\mathfrak{N}_{\varepsilon_2} \equiv \mathfrak{T}_{\alpha-\varepsilon}$.

3. Proof of Theorem 2. Let E be a closed set of measure zero and γ_n the number of dyadic intervals which are not disjoint with E . Hence we have

$$\lim_{n \rightarrow \infty} \text{meas } E_n = \lim_{n \rightarrow \infty} \gamma_n / 2^n = 0.$$

Let m_μ be a dyadic measure which satisfies (2)(i) and (ii). Then m_μ satisfies

$$\begin{aligned} \sum_{k=1}^{2^N-1} \varepsilon_k |\hat{\mu}(k)|^2 &\geq \theta \cdot \sum_{n=0}^N \varepsilon_{2^n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2 \\ &= \theta \sum_{n=0}^N \varepsilon_{2^n} \left(\sum_{k=0}^{2^{n+1}-1} |\hat{\mu}(k)|^2 - \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right) \\ &= \theta \left\{ \sum_{n=0}^N \varepsilon_{2^n} \sum_{k=0}^{2^{n+1}-1} |\hat{\mu}(k)|^2 - \sum_{n=0}^N \varepsilon_{2^n} \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right\} \\ &= \theta \left\{ \sum_{n=0}^{N-1} (\varepsilon_{2^n} - \varepsilon_{2^{n+1}}) \cdot \sum_{k=0}^{2^{n+1}-1} |\hat{\mu}(k)|^2 \right. \\ &\quad \left. - \varepsilon_1 \cdot |\hat{\mu}(0)|^2 + \varepsilon_{2^N} \cdot \sum_{k=0}^{2^{N+1}-1} |\hat{\mu}(k)|^2 \right\}. \end{aligned}$$

From the assumption we have

$$\begin{aligned} 1 = |m_\mu(I_0^0)| &\leq \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)| \\ &\leq \left(\sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^2 \right)^{1/2} \cdot \left(\sum_{(I_n^p E \neq \emptyset)} 1 \right)^{1/2} \leq \left\{ \frac{1}{2^n} \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right\}^{1/2} \cdot \{\gamma_n\}^{1/2} \\ &= \left\{ \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right\}^{1/2} \cdot \left\{ \frac{\gamma_n}{2^n} \right\}^{1/2} = \left\{ \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right\}^{1/2} \cdot \{\text{meas } E_n\}^{1/2}. \end{aligned}$$

Combining the above two inequalities, we obtain

$$\begin{aligned} \sum_{k=0}^{2^N-1} \varepsilon_k |\hat{\mu}(k)|^2 &\geq \theta \left\{ \sum_{n=0}^N \varepsilon_{2^n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2 \right\} \\ &\geq \theta \left\{ \sum_{n=0}^{N-1} (\varepsilon_{2^n} - \varepsilon_{2^{n+1}}) \cdot \frac{1}{\text{meas } E_n} - \varepsilon_1 + \frac{\varepsilon_{2^N}}{\text{meas } E_{N+1}} \right\}. \end{aligned}$$

If E satisfies one of the conditions of Theorem 2, then the last formula tends to infinity. Hence we have

$$\sum_{k=0}^{\infty} \varepsilon_k |\hat{\mu}(k)|^2 = \infty,$$

that is, μ does not belong to $\mathfrak{M}_\varepsilon$. Then E is a set of uniqueness for $\mathfrak{M}_\varepsilon$. The proof is complete.

Theorem 2 solves the problem left open in [3]. Indeed, if E is of α -capacity zero, then E is of α -Hausdorff measure zero (see [2]). Namely, E satisfies

$$\liminf_{n \rightarrow \infty} \gamma_n / 2^{n\alpha} = 0$$

where $\{\gamma_n\}$ is defined as before. Thus, it is easy to see that

$$\limsup_{n \rightarrow \infty} \frac{1}{2^{(1-\alpha)n}} \cdot \frac{1}{\text{meas } E_n} = 0.$$

By Theorem 2, it follows that E is a set of uniqueness for $\mathfrak{U}_\alpha$. In particular, Theorem 2 contains the following corollary.

THEOREM 3. *When $0 < \alpha < 1$, a closed subset is a set of uniqueness for $\mathfrak{U}_\alpha^+$ if and only if it is a set of uniqueness for $\mathfrak{U}_\alpha$.*

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