A SEMIBOUNDED CLOSED SYMMETRIC OPERATOR WHOSE SQUARE HAS TRIVIAL DOMAIN

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ABSTRACT. The existence of closed symmetric operators T such that $D(T^2) = (0)$ was shown by Naimark. This paper gives a simple, explicit construction for such operators and shows that T can be semibounded.

Naimark [2] showed there are closed symmetric operators T on a Hilbert space H such that the domain of T^2 contains only 0. His construction is rather complicated, and the resulting operators are not very explicit. The aim of this note is to give a simple, concrete construction of such operators. Our construction shows that T may be semibounded as well.

We use the machinery of the Cayley transform (see, e.g., [4, p. 502]). Let M and N be closed subspaces of H and let V be an isometry mapping M onto N such that $(V - I)M = D$ is a dense subspace of H . Then $V - I$ is automatically one-to-one, and the relation

$$(1) \quad T = i(V + I)(V - I)^{-1}$$

defines a closed symmetric operator T with dense domain D . Clearly the domain of T^2 reduces to (0) precisely when the ranges of $V + I$ and $V - I$ have trivial intersection. This amounts to requiring that $M \cap N = (0)$. Our task is to exhibit M , N and V satisfying all these conditions.

We take for our Hilbert space $L^2(S)$, where S is the unit circle. For M we take the Hardy class $H^2(S)$. V will be multiplication by a suitable function $\Omega(\theta)$ of modulus one, and $N = VM = \Omega \cdot H^2$. Everything turns on the choice of Ω , which is guided by considering the following well-known result of Szegő [3, 1, p. 53]: If f is a nonzero function in H^2 , then

$$(2) \quad \int_0^{2\pi} \log|f(\theta)| d\theta > -\infty.$$

In particular, f cannot vanish on a set of positive measure.

Define $\Omega(\theta) = \exp(ie^{-1/\theta})$ for $0 < \theta < \pi$, $\Omega(\theta) = -1$ for $\pi \leq \theta \leq 2\pi$. The salient properties of Ω are the following. First, $\Omega(\theta) = -1$ on a set of positive measure; second, $\Omega(\theta) \neq 1$, but

$$(3) \quad \int_0^{2\pi} \log|\Omega(\theta) - 1| d\theta = -\infty;$$

third, the imaginary part of Ω is nonnegative.

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The first property guarantees that $M \cap N = (0)$. Indeed, suppose $f \in H^2$ is such that $\Omega \cdot f$ is also in H^2 , so that $\Omega \cdot f \in M \cap N$. Then $(\Omega + 1)f \in H^2$ but vanishes on a set of positive measure. Hence $(\Omega + 1)f = 0$, which implies f is supported on the arc $\pi \leq \theta \leq 2\pi$, hence $f = 0$.

The second property guarantees that $(V - I)M = (\Omega - 1) \cdot H^2$ is dense. For if $g \in L^2$ is orthogonal to $(\Omega - 1) \cdot H^2$, it follows that $(\Omega - 1)\bar{g} = f$ belongs to H^2 . But (3) implies $\int \log |f| d\theta = -\infty$, so by Szegő's theorem f vanishes, hence g vanishes.

It follows that the operator T defined by (1) does what we want. Note that T is given by multiplication by $F = i(\Omega + 1)(\Omega - 1)^{-1}$ on the domain $D = (\Omega - 1) \cdot H^2$. Moreover the function F is nonnegative because $\text{Im}(\Omega) \geq 0$. Hence the closed symmetric operator T is nonnegative.

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