A NOTE ON CHAINS OF OPEN SETS

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ABSTRACT. We consider some questions concerning the nature and size of chains of open sets in Hausdorff spaces. The following results are obtained.

THEOREM 1. For every cardinal κ there exists a space X in which all discrete subsets have cardinality at most κ and which contains a chain of $(2^\kappa)^+$ open sets.

THEOREM 2. If X is regular and contains a chain of $(2^\kappa)^+$ open sets, then $X \times X$ contains a discrete subset of cardinality κ^+ .

THEOREM 3. Let $M(X)$ denote the set of all maximal chains of open subsets of X endowed with the Tychonoff topology. (i) $|M(X)| \leq 2^{w(X)}$, and (ii) $\psi(M(X)) \leq w(X)$. Here $w(X)$ denotes the weight of the space X and $\psi(M(X))$ denotes the pseudocharacter of the space $M(X)$.

1. Introduction. We will consider some questions concerning the nature and size of chains of open sets in Hausdorff topological spaces. All topological spaces considered are assumed to be Hausdorff. The set of all open subsets of the space X will be denoted by $T(X)$. Our terminology and notation for set-theoretic and topological concepts is standard; a basic reference is [8]. In particular, if κ is a cardinal number then κ^+ denotes the first cardinal greater than κ . All cardinals discussed are assumed to be infinite. The cardinality of a set S is denoted by $|S|$, and the power set of S is denoted by $P(S)$.

Recall that if $(P, \leq)$ is a partially ordered set then elements a and b of P are said to be *comparable* if either $a \leq b$ or $b \leq a$; otherwise a and b are said to be *incomparable*. A subset S of P is called a *chain* if any two elements of S are comparable, and S is called an *antichain* if no two elements of S are comparable. (Note that our meaning of the term "antichain" differs from that commonly used by logicians.) Thus when we refer to a chain of open sets in a space X , we are speaking of a chain in the partially ordered set $(T(X), \subseteq)$.

There are three natural ways in which chains of open sets are encountered in topological spaces. Probably the most familiar is the following: Let X be a space and κ a cardinal number. If S is a subspace of X which contains no dense subset of cardinality $< \kappa$, then we can inductively choose points x_α in S for $\alpha < \kappa$ such that $x_\alpha \notin \text{cl}\{x_\beta : \beta < \alpha\}$ (the symbol $\text{cl } T$ denotes the closure of T in X). The result is a chain of κ open sets $\{G_\alpha : \alpha < \kappa\}$ where $G_\alpha = X - \text{cl}\{x_\beta : \beta < \alpha\}$. Similarly, if X contains a subspace S which has an open cover containing no subcover by less than

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κ sets, then we can inductively choose open sets H_α in X and points x_α in S for $\alpha < \kappa$ such that H_α is a member of the open cover containing x_α and $x_\alpha \notin H_\beta$ for $\beta < \alpha$. This results in a chain of κ open sets $\{G_\alpha: \alpha < \kappa\}$ where $G_\alpha = \bigcup \{H_\beta: \beta \leq \alpha\}$. In these cases we either get a *well-ordered or dually well-ordered chain of open sets*.

A second way to form chains is the following. Let $\{G_i: i \in I\}$ be any indexed family of open sets in a space X . If A and B are subsets of the index set I and $A \subseteq B$, then $\bigcup \{G_i: i \in A\} \subseteq \bigcup \{G_i: i \in B\}$, so any chain γ of subsets of I gives rise to a chain of open sets $\{H_A: A \in \gamma\}$ where $H_A = \bigcup \{G_i: i \in A\}$. In order to ensure that this gives rise to a chain of open sets as large as γ , we want to require that $H_A \subseteq H_B$ only if $A \subseteq B$. This is equivalent to requiring that $G_i \not\subseteq \bigcup \{G_j: j \neq i\}$ for each i . In this case for each i in I there exists a point $x_i \in G_i - \bigcup \{G_j: j \neq i\}$. Let $S = \{x_i: i \in I\}$. Since $G_i \cap S = \{x_i\}$, S is a discrete subset of X . Conversely, if we begin with a discrete subset S of X , then for each s in S there is an open set G_s in X such that $G_s \cap S = \{s\}$. The family of open sets $\{G_s: s \in S\}$ clearly has the property that, for subsets A, B of S , $\bigcup \{G_s: s \in A\} \subseteq \bigcup \{G_s: s \in B\}$ if and only if $A \subseteq B$, so we can obtain chains of open sets in X from chains of subsets of S , that is, from chains in $P(S)$. We refer to these chains of open sets as *chains arising from discrete subsets*.

A third, and probably the most simple, way in which we may encounter chains of open sets is the following: Suppose $(T, \leq)$ is a totally ordered set endowed with the order topology (having all open intervals as a basis). For each t in T we let (t, ∞) and $(-\infty, t)$, respectively, denote the sets $\{x \in T: x > t\}$ and $\{x \in T: x < t\}$. The family $\{(t, \infty): t \in T\}$ is a chain of open sets in T (as is the family $\{(-\infty, t): t \in T\}$). Thus if X is a space, then any continuous function f from X into a totally ordered space $(T, \leq)$ gives rise to a chain of open sets $\{G_t: t \in T\}$ where $G_t = f^{-1}(t, \infty)$ for each t in T . In order to ensure that this gives rise to a chain of open sets as large as T , we clearly want every nonempty interval (t_1, t_2) in T to contain $f(x)$ for some element x of X . For, if f has this property, let $\kappa = |T|$ and let T_1 be a subset of T of cardinality κ such that no element of T_1 is the successor in T of any other element of T_1 . Then the sets G_t for $t \in T_1$ are all distinct. In other words, if the image of X is dense in T then we get a chain of open sets in X as large as T . We refer to such chains as *arising from totally ordered spaces*.

We may now inquire into the extent to which these three processes account for the sizes of chains of open sets in Hausdorff spaces in general. We will discuss chains arising from discrete sets in §2 so we confine our comments here to well-ordered and dually well-ordered chains and chains arising from totally ordered spaces.

The *Lindelöf number*, *hereditary Lindelöf number*, *density character* and *hereditary density character* of a space X are denoted by $L(X)$, $hL(X)$, $d(X)$ and $hd(X)$ respectively; definitions are given in [8]. As discussed in [8], and as indicated above, we have

$$hL(X) = \sup\{\kappa: X \text{ contains a well-ordered chain of } \kappa \text{ open sets}\},$$

$$hd(X) = \sup\{\kappa: X \text{ contains a dually well-ordered chain of } \kappa \text{ open sets}\}.$$

Now let X be a Hausdorff space. As noted in Theorem 2.11 of [9], we have $|T(X)| \leq |X|^{hd(X)}$, since every closed subset of X is the closure of a subset of X

having cardinality $\leq \text{hd}(X)$. But $|X| \leq 2^{\text{hd}(X)}$ (see [8]) so we have $|T(X)| \leq 2^{\text{hd}(X)\text{hd}(X)}$. In particular, if $|T(X)| > 2^\kappa$ then X contains either a well-ordered or dually well-ordered chain of open sets which has cardinality $> \kappa$. Simple examples like the real line show that the existence of a chain of κ open sets need not imply the existence of a well-ordered or dually well-ordered chain of the same size.

One immediate consequence of the inequality $|T(X)| \leq 2^{\text{hd}(X)\text{hd}(X)}$ is that $|T(X)| \leq 2^{l(T(X))}$. Here $l(T(X))$ denotes the *length* of $T(X)$; where, for any partially ordered set $(P, \leq)$,

$$l(P) = \sup\{\kappa: P \text{ contains a chain of cardinality } \kappa\}.$$

It is natural to inquire as to whether the above inequality can be strengthened to $|T(X)| = l(T(X))$. More specifically, if X is a Hausdorff space must $T(X)$ contain a chain of cardinality $|T(X)|$? Although it is easy to give *consistent* counterexamples to this question, the author has been unable to construct a counterexample just within ZFC. For example, if X is a discrete space of cardinality κ , then $T(X)$ is just $P(X)$, the power set of X . It follows from results in [10] that it is consistent that $P(X)$ contains no chains of size 2^κ . Another example is any hereditarily separable, regular space of cardinality $> 2^\omega$, such as the one constructed in [5]. It can be shown that such a space has no chain of open sets of cardinality $> 2^\omega$. On the other hand, for regular spaces at least, it may be consistent that $|T(X)| = l(T(X))$; the author has been unable to settle this question.

As for chains arising from totally ordered spaces, it will now be shown that if we are willing to replace the property of continuity by semicontinuity, then every chain of open sets arises in this way. If X is a space and $(T, \leq)$ is a totally ordered space, recall that a function f from X to T is said to be *upper semicontinuous* if $\{x \in X: f(x) < t\}$ is open in X for each t in T , and *lower semicontinuous* if $\{x \in X: f(x) > t\}$ is open in X for each t in T . If f is either upper or lower semicontinuous we simply say that f is *semicontinuous*. We have the following result.

1.1. THEOREM. *Let X be a Hausdorff space and κ an infinite cardinal. The following are equivalent:*

- (i) X contains a chain of κ open sets;
- (ii) *there exists a totally ordered space $(T, \leq)$ of cardinality $\geq \kappa$ and a semicontinuous function $f: X \rightarrow T$ such that $f(X)$ is dense in T .*

PROOF. The preceding discussion shows that (ii) implies (i); just note that only semicontinuity of the function is needed.

Conversely, assume that (i) holds. Let γ be a chain of open sets in X with $|\gamma| = \kappa$. Let m be a maximal chain of open sets in X which contains γ . Thus $|m| \geq \kappa$. Note that, by maximality, if s is any subset of m then $\bigcup s \in m$. We take the totally ordered space $(T, \leq)$ to be $(m, \subseteq)$ and define $f: X \rightarrow m$ by

$$f(x) = \bigcup \{G \in m: x \notin G\}.$$

For each G in m we have $\{x \in X: f(x) \not\subseteq G\} = G$, so f is (upper) semicontinuous, and $f(X)$ is dense in m : for let (G_1, G_2) be any nonempty open interval in m . Let G be a member of m such that $G_1 \subset G \subset G_2$. Let x be an element of X such that

$x \in G_2 - G$. Then $f(x) \supseteq G$ because $x \notin G$. Also, since $x \notin f(x)$ and $x \in G_2$, it follows that G_2 is not contained in $f(x)$. Therefore we must have $f(x) \subsetneq G_2$ (since m is a chain). Thus we have $G \subseteq f(x) \subsetneq G_2$ and so $f(x) \in (G_1, G_2)$. This shows that $f(X)$ is dense in m .

We note that semicontinuity cannot be replaced by the stronger condition of continuity in 1.1(ii). For example, let $X = \{0, 1\}^\kappa$. Since X contains no uncountable family of pairwise disjoint open sets, neither does any continuous image of X . Thus if there exists a continuous map from X onto a dense subset of an ordered space T , then T contains no uncountable family of disjoint open sets. It is well known (see [8]) that such a totally ordered set T has cardinality at most 2^ω . However, it is easy to see that $\{0, 1\}^\kappa$ contains a chain of κ open sets. For example, for each $\alpha < \kappa$, let $G_\alpha = \{f \in \{0, 1\}^\kappa : f(\xi) = 1 \text{ for some } \xi < \alpha\}$. Then $\{G_\alpha : \alpha < \kappa\}$ is a chain. In fact, with a bit more work it can be seen that this space contains a chain of size κ^+ : for $\{0, 1\}^\kappa$ has a discrete subset of cardinality κ and, hence, will contain chains of open sets as large as the chains in $P(\kappa)$, the power set of κ (chains arising from discrete sets). Since $P(\kappa)$ contains a chain of cardinality κ^+ (see [10]), our statement follows. In particular, for $\kappa > 2^\omega$ the space $\{0, 1\}^\kappa$ contains a chain of κ open sets but cannot be mapped continuously onto a dense subset of any totally ordered space $(T, \leq)$ with $|T| \geq \kappa$.

2. Chains of open sets and discrete subsets. Let X be a Hausdorff space. As in [8] we let $s(X)$ denote the *spread* of X :

$$s(X) = \sup\{\kappa : X \text{ contains a discrete subset of cardinality } \kappa\}.$$

Many relationships are known between the cardinality of $T(X)$ and that of discrete subsets of X . For example, Theorem 2.21 of [9] states that, for any Hausdorff space X , $|T(X)| \leq 2^{s(X)}$. In [7] a much stronger relation between the cardinality of $T(X)$ and that of discrete subsets is derived for spaces of singular cardinality, which implies that, assuming the generalized continuum hypothesis (GCH), if X is a Hausdorff space of singular cardinality, then X contains a discrete subset of cardinality $|X|$. In this case, $|T(X)| = 2^{|X|} = 2^{s(X)}$. This statement need not hold when $|X|$ is regular, as is shown by the hereditarily separable, compact space of cardinality 2^c constructed in [5].

Let us now consider the relation between the sizes of chains in $T(X)$ and the cardinality of discrete subsets of X . As pointed out in §1, if D is a discrete subset of X then $T(X)$ contains chains as large as those of $P(D)$. Now $P(D)$ always contains a chain of cardinality $|D|^+$ (for example, see [10]). Thus, assuming GCH, $P(D)$ contains a chain of size $2^{|D|}$ (it is also consistent, as mentioned in the previous section, that $P(D)$ contains no chains of cardinality $2^{|D|}$; see [10]). Thus if X contains a discrete subset D with $|D| > \kappa$, it follows that $T(X)$ contains a chain of cardinality $> \kappa^+$. Assuming GCH this result can be stated as follows: If $s(X) > \kappa$ then $l(T(X)) > 2^\kappa$. Let us now consider the converse situation. Suppose $T(X)$ contains a chain of cardinality $> 2^\kappa$. Must X contain a discrete subset of cardinality $> \kappa$? We will now show that the answer is no in general. First we establish the following lemma, where $w(X)$ denotes the *weight* of the space X —the least cardinal κ such that X has a basis of cardinality κ .

2.1. LEMMA. *Let X be a Hausdorff space with topology $T(X)$. Let $w(X) = \kappa$ and let Δ be a family of subsets of X which is closed under finite intersections and which contains no antichains of cardinality κ^+ . Let Y denote the set X together with the topology τ having as subbasis the family $T(X) \cup \Delta$. Then $s(Y) \leq \kappa$.*

PROOF. We may assume without loss of generality that $\emptyset \in \Delta$ and $X \in \Delta$. Let b be a basis for X with $|b| = \kappa$. Since Δ is closed under finite intersections, sets of the form $G \cap S$, where G is in b and S is in Δ , form a basis for the topology τ . For the sake of contradiction, suppose the space Y contains a discrete subset D with $|D| = \kappa^+$. For each x in D , there exist sets G_x in b and S_x in Δ such that $(G_x \cap S_x) \cap D = \{x\}$. Since $|b| = \kappa$ there exists a set $G \in b$ and a subset D_1 of D with $|D_1| = \kappa^+$ such that $G_x = G$ for all $x \in D_1$. Therefore the sets S_x for x in D_1 are all distinct. By our assumption on Δ the family $\{S_x: x \in D_1\}$ is not an antichain, so there exist distinct elements x and y of D_1 with $S_x \subset S_y$. For such x and y we have $x \in G_x \cap S_x = G \cap S_x \subseteq G \cap S_y = G_y \cap S_y$, which contradicts the fact that $(G_y \cap S_y) \cap D = \{y\}$. Therefore Y contains no discrete subset of cardinality κ^+ so $s(Y) \leq \kappa$.

2.2. THEOREM. *Let κ be an infinite cardinal. Then there exists a Hausdorff space Y which contains a chain of $(2^\kappa)^+$ open sets and in which all discrete subsets have cardinality at most κ .*

PROOF. Let X be any Hausdorff space of weight κ and cardinality 2^κ , for example the space $\{0, 1\}^\kappa$. As mentioned above, $P(X)$ contains a chain Δ of cardinality $(2^\kappa)^+$. The preceding lemma can now be applied to the space X and Δ to yield the desired space Y .

REMARK. We note that the space Y obtained in 2.2 is in general not regular. The author has been unable to find a regular space Y having the above properties. We do not know whether such a regular space can be constructed in ZFC or whether it is consistent with ZFC that any regular space containing a chain of more than 2^κ open sets must contain a discrete subset of cardinality κ^+ . However we can prove the following result for regular spaces.

2.3. THEOREM. *Let X be a regular space which contains a chain of open sets which has cardinality $> 2^\kappa$. Then $X \times X$ contains a discrete subset of cardinality κ^+ .*

PROOF. We will show that $hL(X) > \kappa$ and $hd(X) > \kappa$. The first implies that X contains a subset $S = \{x_\alpha: \alpha < \kappa^+\}$ and open sets $\{G_\alpha: \alpha < \kappa^+\}$ such that $x_\alpha \in G_\alpha$ and $x_\beta \notin G_\alpha$ if $\beta > \alpha$. The second implies that X contains a subset $T = \{y_\alpha: \alpha < \kappa^+\}$ and open sets $\{H_\alpha: \alpha < \kappa^+\}$ such that $y_\alpha \in H_\alpha$ and $y_\beta \notin H_\alpha$ if $\beta < \alpha$. As noted in [11] this yields a discrete subset $D = \{(x_\alpha, y_\alpha): \alpha < \kappa^+\}$ in the square $X \times X$.

Thus let γ be a chain of open subsets of the regular space X with $|\gamma| = (2^\kappa)^+$. We now make use of the Erdős-Rado partition relation $a^+ \rightarrow (a^+, b)$, where a is any infinite cardinal and b is the smallest cardinal for which $a^b > a$ (see [4]). This implies, in particular, that $(2^\kappa)^+ \rightarrow ((2^\kappa)^+, \kappa^+)$. Now suppose γ contains no subfamily of size $(2^\kappa)^+$ which is either well-ordered or dually well-ordered under inclusion. Let $\leq$ be any well-ordering of γ . By placing the pairs $\{C, D\}$ in box 1 if

the inclusion order $\subseteq$ agrees with the well-ordering $\leq$ on $\{C, D\}$, in box 2 if the two orderings are opposite on $\{C, D\}$, applying the partition relation $(2^\kappa)^+ \rightarrow ((2^\kappa)^+, \kappa^+)$, and then doing the same thing with boxes 1 and 2 interchanged, we see that γ must contain a subfamily of size κ^+ which is well-ordered (by inclusion) and one of the same cardinality which is dually well-ordered. This means that $\text{hL}(X) > \kappa$ and $\text{hd}(X) > \kappa$ is desired.

Thus we may assume that γ either contains a well-ordered or a dually well-ordered (by inclusion) subfamily of cardinality $(2^\kappa)^+$. First suppose γ contains a well-ordered subfamily γ_1 with $|\gamma_1| = (2^\kappa)^+$. Then $w(X) \geq \text{hL}(X) \geq |\gamma_1| > 2^\kappa$. But X is regular so $w(X) \leq 2^{d(X)}$. Therefore we must have $d(X) > \kappa$ which means X contains a dually well-ordered chain of κ^+ open sets. Thus, in this case we have $\text{hd}(X) > \kappa$ and $\text{hL}(X) > \kappa$ (in fact $\text{hL}(X) > 2^\kappa$), as desired. Secondly, suppose γ contains a dually well-ordered subfamily having cardinality $(2^\kappa)^+$. This clearly implies that $|X| \geq (2^\kappa)^+$. Since $|X| \leq 2^{\text{hL}(X)}$ (see [8]), it follows that $\text{hL}(X) > \kappa$. Since we also have $\text{hd}(X) > \kappa$ (in fact $\text{hd}(X) > 2^\kappa$), again the result follows.

REMARKS. (i) It is interesting to compare 2.3 with a result in [2] concerning zero-dimensional compact spaces: if X is a compact zero-dimensional space which contains a chain of κ clopen sets, then $X \times X$ contains a discrete subset of cardinality κ .

(ii) The author has been unable to determine whether 2.3 holds for all Hausdorff spaces. Certainly the proof as given cannot be extended to Hausdorff spaces since use was made of the fact that $w(X) \leq 2^{d(X)}$, which does not hold for Hausdorff spaces in general.

3. A result on bases in partially ordered sets and two applications to maximal chains of open sets. In this section we will establish a result about partially ordered sets which may be of independent interest and which has two consequences for maximal chains of open sets.

Let $(P, \leq)$ be a partially ordered set. If S is a subset of P , $p \in P$, and p is the least upper bound of S in P we write $p = \bigvee S$. A subset B of P is called a *base* for P if for every element p of P there exists a subset S of B such that $p = \bigvee S$. In this case we also say that B *generates* P . We say that P is κ -generated if P has a base B such that $|B| \leq \kappa$. If A is a subset of P we let $A^\Sigma = \{p \in P: \text{there exists a subset } S \text{ of } A \text{ such that } p = \bigvee S\}$. Thus B is a base for P if and only if $B^\Sigma = P$. More generally if $(P, \leq)$ is a partially ordered set and $Q \subseteq P$, we say that Q is κ -generated in P if there exists a subset B of Q with $|B| \leq \kappa$ and $Q \subseteq B^\Sigma$.

Note that if X is a topological space then $(T(X), \subseteq)$ is κ -generated if and only if $w(X) \leq \kappa$.

We note that a subset Q of a κ -generated partially ordered set P is not necessarily κ -generated in P . For example, if Q is an antichain in P , then clearly the only subset B of Q for which $Q \subseteq B^\Sigma$ is $B = Q$. Thus if Q is an antichain of cardinality κ^+ then Q is not κ -generated in P . An example of a κ -generated partially ordered set which contains an antichain of cardinality κ^+ is $P(X)$ where X is any set of cardinality κ . (It is well known that $P(X)$ contains an antichain of cardinality $2^{|X|}$; one way to see this is as follows: let f be a bijection from $X \times X$ to X . Then the family

$\{f[A \times (X - A)]: A \in P(X)\}$ is an antichain.) It is tempting to conjecture that in a κ -generated partially ordered set, any subset which contains no antichain of cardinality κ^+ is itself κ -generated. However, even for the case $\kappa = \omega$ this conjecture is independent of the usual axioms of set theory. We will return to this point later. We can, however, establish the following sufficient condition for a subset of a κ -generated partially ordered set to be κ -generated.

3.1. LEMMA. *Let $(P, \leq)$ be a κ -generated partially ordered set. Let Q be a subset of P such that every subset of Q of cardinality $> \kappa$ contains an increasing ω -sequence. Then Q is κ -generated in P .*

PROOF. Let B be a base for P of cardinality $\leq \kappa$. We assume Q is not κ -generated in P , and we will reach a contradiction.

We inductively select, for each $\alpha < \kappa^+$, elements q_α and b_α in Q and B , respectively, as follows: Let q_0 be any element of Q and let b_0 be any element in B with $b_0 \leq q_0$ (since $P \subseteq B^\Sigma$ such an element b_0 exists). Now suppose $\alpha < \kappa^+$ and that for every $\beta < \alpha$ we have selected $q_\beta \in Q$ and $b_\beta \in B$. Let $Q_\alpha = \{q_\beta: \beta < \alpha\}$. Then $|Q_\alpha| \leq \kappa$ so, by our assumption, $Q \not\subseteq Q_\alpha^\Sigma$. Thus there exists an element $q_\alpha \in Q - Q_\alpha^\Sigma$. Therefore q_α is not the supremum (in P) of any subset of Q_α . In other words, q_α is not the supremum of the set $R_\alpha = \{q_\beta: \beta < \alpha \text{ and } q_\beta < q_\alpha\}$. Since $P \subseteq B^\Sigma$ there is a subset B_α of B such that $q_\alpha = \bigvee B_\alpha$. There exists an element b_α in B_α such that $b_\alpha \not\leq q$ for all q in R_α . (If every element of B_α were dominated by some element of R_α then any upper bound of R_α would be an upper bound of B_α and so would be $\geq q_\alpha$. But this would imply that $q_\alpha = \bigvee R_\alpha$.) This completes the inductive construction. The result is a sequence of pairs (q_α, b_α) for $\alpha < \kappa^+$ such that:

- (i) $q_\alpha \in Q$ and $b_\alpha \in B$ for all $\alpha < \kappa^+$;
- (ii) $b_\alpha \leq q_\alpha$ for all $\alpha < \kappa^+$;
- (iii) if $\beta < \alpha < \kappa^+$ and $q_\beta \leq q_\alpha$, then $b_\alpha \not\leq q_\beta$.

Now since $|B| \leq \kappa$ the pigeon-hole principle implies there exists a subset S of κ^+ of cardinality κ^+ and an element b of B such that $b_\alpha = b$ for all $\alpha \in S$. By our assumption on Q , the set $T = \{q_\alpha: \alpha \in S\}$ contains an increasing ω -sequence $q_{\alpha_1} < q_{\alpha_2} < \dots < q_{\alpha_n} < \dots$ where $\{\alpha_n: n \in \omega\} \subseteq S$. Now, since there exists no decreasing sequence of ordinals, there must exist integers m and n with $m < n$ and $\alpha_m < \alpha_n$. But since $q_{\alpha_m} < q_{\alpha_n}$ we have by (iii) that $b_{\alpha_n} \not\leq q_{\alpha_m}$. But $b_\alpha = b$ for all α in S , which means that $b \not\leq q_{\alpha_m}$, which contradicts (ii).

3.2. COROLLARY. *Suppose $(P, \leq)$ is κ -generated. Let Q be a subset of P such that every subset of Q of cardinality κ^+ contains a chain of cardinality κ^+ . Then Q is κ -generated in P . In particular, every chain in P is κ -generated.*

PROOF. In light of 3.1 it is sufficient to show that every chain in P of cardinality κ^+ contains an increasing ω -sequence. Thus, let R be a chain in P of cardinality κ^+ . Since P has a base of cardinality $\leq \kappa$, it is clear that P contains no dually well-ordered subset of cardinality κ^+ . If we now well-order R and partition the pairs of elements of R into two boxes as in the proof of 2.3 above, we find, on applying the partition relation $\kappa^+ \rightarrow (\omega, \kappa^+)$, that R contains an infinite subset which is

well-ordered by $\leq$. Since any infinite well-ordered set contains an increasing ω -sequence, the result follows.

3.3. COROLLARY. *Let X be a Hausdorff space and let $\kappa = w(X)$. Then every chain of open subsets of X is κ -generated.*

We note there is a simpler, direct proof that a *maximal* chain of open sets in a space of weight κ is κ -generated: let b be a basis for X with $|b| = \kappa$. If m is any maximal chain of open sets in X , we define, for each B in b ,

$$G_B = \text{int} \left(\bigcap \{ G \in m : B \subseteq G \} \right),$$

where int denotes the interior in X . It is straightforward to show that each set G_B is in m and every member of m is a union of a subfamily of $\{G_B : B \in b\}$ (see also Lemma 4 in [1]).

Now, let X be a Hausdorff space and let $M(X)$ denote the set of all maximal chains of open subsets of X . That is $M(X)$ is the set of all maximal chains of $T(X)$.

3.4. THEOREM. *Let X be a Hausdorff space. Then $|M(X)| \leq 2^{w(X)}$.*

PROOF. Let m be a maximal chain in $T(X)$. By 3.3, m contains a subfamily b of cardinality $\leq w(X)$ such that $m \subseteq b^\Sigma$. But by maximality the union of any subfamily of m belongs to m , so $m = b^\Sigma$. Since there are at most $|T(X)|^{w(X)}$ such families b of cardinality $\leq w(X)$, it follows that there are at most $|T(X)|^{w(X)}$ maximal chains in $T(X)$. But $|T(X)| \leq 2^{w(X)}$, so $|M(X)| \leq (2^{w(X)})^{w(X)} = 2^{w(X)}$.

If X is a discrete space then $T(X) = P(X)$, and $w(X) = |X|$. Now $P(X)$ clearly has at least $2^{|X|}$ maximal chains, since $P(X)$ has an antichain of that size, so as a corollary to 3.4 we obtain the following well-known result.

3.5. COROLLARY. *Let X be a set. Then $P(X)$ contains exactly $2^{|X|}$ maximal chains.*

A second application of 3.3 concerns the set of maximal chains viewed as a topological space. In general, for any partially ordered set $(P, \leq)$, we let $c(P)$ denote the set of all chains in P , and we let $m(P)$ denote the set of all maximal chains in P . Since $c(P)$ is a subset of the power set of P , it can be endowed with the Tychonoff product topology as a subspace of $\{0, 1\}^P$. This is the topology on $c(P)$ having as a subbasis all sets of the form $a(x) = \{\gamma \in c(P) : x \in \gamma\}$ and $b(x) = \{\gamma \in c(P) : x \notin \gamma\}$ for x in P . $c(P)$ is a compact Hausdorff space. Of course $m(P)$ carries the subspace topology.

Many properties of these spaces of chains will be discussed in [3]. Here we only wish to mention how 3.3 can be used to obtain a result about the space $M(X) = m(T(X))$. That is, for a Hausdorff space X , we consider the partially ordered set $(T(X), \subseteq)$ and the space $m(T(X))$ of maximal chains of $T(X)$ as a subspace of the space $c(T(X))$ of all chains of $T(X)$.

Recall that the *pseudocharacter* of a space Y , denoted $\psi(Y)$, is the least cardinal κ such that every point of Y is equal to the intersection of a family of at most κ open sets.

3.6. THEOREM. $\psi(M(X)) \leq w(X)$.

PROOF. Let $\gamma \in M(X)$ and let $\kappa = w(X)$. Then, as in the proof of 3.4, γ contains a subfamily b of cardinality $\leq \kappa$ such that $\gamma = b^\Sigma$. In the space $M(X)$ this implies that $\bigcap \{a(G) \cap M(X) : G \in b\} = \{\gamma\}$, since any maximal chain which belongs to $a(G)$ for each G in b must contain b , and, therefore, by maximality, must contain $b^\Sigma = \gamma$. Thus the point γ is equal to the intersection of at most $|b|$ open sets in $M(X)$ so $\psi(M(X)) \leq \kappa$.

REMARK. We now return to a comment made before the statement of 3.1, concerning κ -generated partially ordered sets and subsets which contain no antichain of cardinality κ^+ . Whether such subsets are κ -generated, even in the case when $\kappa = \omega$, is independent of the usual axioms for set theory. This follows from results in [1], where the following proposition is shown to be consistent with the axioms of set theory: every uncountable subset of $P(\omega)$ contains an uncountable chain or an uncountable antichain. Now every ω -generated partially ordered set P is order embedded in $(P(\omega), \subseteq)$. For, if B is a countable base for P , and if for each x in P we let $f(x) = \{b \in B : b \leq x\}$, then f is an order embedding of P into $(P(B), \subseteq)$, which is isomorphic to $(P(\omega), \subseteq)$. Therefore in conjunction with 3.2 above, the above consistency result from [1] shows it is consistent that, in an ω -generated partially ordered set P , any subset which contains no uncountable antichains is itself ω -generated. (This consistency also follows, in conjunction with 3.1 above, from a result, stated in [1], due to K. Kunen: Martin's Axiom, together with the negation of the continuum hypothesis, implies that any uncountable subset of $P(\omega)$ with no uncountable antichains must contain both increasing and decreasing ω -sequences under inclusion.)

On the other hand, as stated in [1], and due independently to K. Kunen and J. Baumgartner, assuming the continuum hypothesis there exists a sequence $\sigma = \{X_\alpha : \alpha < \omega_1\}$ of subsets of ω such that σ contains no uncountable antichains (under inclusion) and such that if $\alpha < \beta < \omega_1$ then $X_\alpha \not\subseteq X_\beta$. This last property clearly implies that σ is not ω -generated, although it is a subset of the ω -generated partially ordered set $(P(\omega), \subseteq)$. This proves the independence statement made above.

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