A HILBERT SPACE LIMIT FOR THE ITERATED HYPERSPACE FUNCTOR¹

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ABSTRACT. Let X be a nondegenerate metric Peano continuum and let $P(X)$ be the hyperspace of closed, nonvoid subsets of X equipped with the Hausdorff metric. Then the inclusion of X into $P(X)$ as the single element sets is an isometry and we have a direct system $X \rightarrow P(X) \rightarrow P(P(X)) \rightarrow \cdots$ of isometric inclusions. Let X' be the metric direct limit and X^* be its completion. We prove that the pair (X^*, X') is homeomorphic to the pair (l^2, l^2_σ) , where l^2_σ is the linear span in l^2 of the Hilbert cube.

1. Introduction. If one has a set S , one may form the set $P(S)$ of all subsets of S and then $P(P(S))$ and so on *ad infinitum*. One also has the inclusion of S into $P(S)$ as the one-element sets. There are various ways of topologizing different collections of subsets of a topological space and of taking limits of sequences of spaces and mappings. What sorts of spaces can one obtain from these ideas, and how do they relate to other better-known spaces?

In this paper we investigate one of the most natural examples of such constructions and show that for a well-known class of spaces (Peano continua) the limit in question is always homeomorphic to the separable Hilbert space l^2 , thus answering positively a conjecture of the second author [12].

Our work consists of showing that the topological characterization of l^2 recently derived by the first author [11] is satisfied.

Our construction is as follows: let X be a compact metric space with metric d , and let $\exp(X)$ be the "hyperspace" of all nonvoid, closed subsets of X topologized by the Hausdorff metric, also denoted by d (for $A, B \in \exp(X)$),

$$d(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(a, b) \right\}.$$

Now $(\exp(X), d)$ is again a compact metric space, so we may form inductively $\exp^n(X) = \exp(\exp^{n-1}(X))$, with Hausdorff metric d . Using the inclusion $X \rightarrow \exp(X)$ given by $x \rightarrow \{x\}$, we have a direct system

$$X \rightarrow \exp(X) \rightarrow \exp^2(X) \rightarrow \cdots$$

and can form the set-theoretic direct limit, X' . Each inclusion is an isometric embedding, so there is induced a metric on X' which we also call d . If X is not

Received by the editors October 25, 1982.

1980 *Mathematics Subject Classification*. Primary 54B20, 54B25; Secondary 57N20.

Key words and phrases. Hyperspace, direct limit, Peano continuum, Hilbert cube, Hilbert space.

¹Given at 1978 Hungarian Topology Conference, Budapest.

²Supported by the U.S. and Polish Academies of Science.

discrete, then X' is not complete, and we form the completion X^* , with metric d . We note that this construction $X \rightarrow X^*$ is functorial on the category of compact metric spaces and continuous mappings, yielding complete separable metric spaces and uniformly continuous mappings; therefore, the topological types of X' and X^* are independent of the metric chosen for X . *Our main result is that if X is a Peano continuum, then X^* is homeomorphic to l^2 ; we also obtain the result that X' is homeomorphic to l^2_σ , the linear span in l^2 of the Hilbert cube*

$$Q = \{x = (t_1, t_2, \dots) \in l^2 \mid |t_i| \leq 1/i\}.$$

We remark that the topological direct limit X_∞ of the above sequence (using the “weak” topology for X') also yields, for Peano continua, a well-known locally convex topological vector space: (l^2, bw) , i.e., the set l^2 equipped with the bounded-weak topology. (This observation is due to R. E. Heisey [8], who has studied manifolds modelled on (l^2, bw) intensively; for, the pair $(\exp^{n+1}(X), \exp^n(X))$ is homeomorphic as a pair to $(Q \times [0, 1], Q \times \{0\})$.)

2. Notation and definitions. A Hilbert cube is any space homeomorphic to Q above. Functions are not necessarily continuous; mappings are continuous. A mapping $f: A \rightarrow B$ may be approximated by maps $g: A \rightarrow B$ with some property if for any open cover $\mathcal{U}$ of B there is a g such that for each $a \in A$ there is a $U \in \mathcal{U}$ containing both $f(a)$ and $g(a)$; in this case, we say that g is $\mathcal{U}$ -close to f . ANR (AR) means here Absolute Neighborhood Retract (Absolute Retract) for the class of metric spaces, and is used in the sense of extensors. A subset A of an ANR B is a Z -set (has Property Z in B) if it is closed and if each map $f: I^n \rightarrow B$, $n \in N$, may be approximated by maps missing A . (See [4 and 6] for general discussions.) We denote the positive integers by N . All metrics will be denoted by d . A metric d for a space X is *convex* if for each two points $x, y \in X$ there is a (not necessarily unique) midpoint $z \in X$ with $d(x, z) = d(z, y) = d(x, y)/2$. Every Peano continuum admits a convex metric [5, 9], and we assume that X is always a Peano continuum and that d is convex. From the convexity, it follows immediately that between each pair of points x, y in X there is an isometric embedding of the interval $[0, d(x, y)]$, which we call a *geodesic*.

We think of X as $\exp^0(X)$ and identify $\exp^n(X)$ with its images in $\exp^{n+m}(X)$, X' , and X^* . When necessary, we use “ i ” to denote the natural inclusions of these spaces in each other. Points of X' or X^* will be denoted by x ; points of $\exp^n(X)$, by A . When it is necessary to think of a point $x \in X'$ as an equivalence class, we write $A \in x$, etc.

There is a naturally defined left inverse u_n to the inclusion $i: \exp^n(X) \rightarrow \exp^{n+1}(X)$, for $n > 0$. This is derived from the set-theoretic operation of “union” as follows:

$$u(A) = \bigcup \{B \in \exp^n(X) \mid B \in A\} \in \exp^n(X).$$

(Thus, $u(\{B\}) = B$.) This extends to a function $u_n: X' \rightarrow \exp^n(X)$ given by

$$u_n(x) = \bigcup \{B \in \exp^n(X) \mid B \in B_1 \in B_2 \in \dots \in B_i = A \in x\}$$

for some sequence $\{B_i \in \exp^{n+i}(X)\}_{i=1}^\infty$ and some $A \in x$.

Each $u_n: \exp^{n+1}(X) \rightarrow \exp^n(X)$ is nonexpansive, i.e.,

$$d(u_n(A), u_n(A')) \leq d(A, A'),$$

so $u_n: X' \rightarrow \exp^n(X)$ is uniformly continuous and induces a nonexpansive mapping $u_n: X^* \rightarrow \exp^n(X)$.

If $A \in \exp(Y)$ and $t \geq 0$, $N_t(A) \in \exp(Y)$ denotes the closed t -neighborhood of A in Y , i.e.,

$$N_t(A) = \{y \in Y \mid \text{for some } a \in A, d(a, y) \leq t\}.$$

We note that because (X, d) is a Peano continuum with a convex metric, N_t is continuous and $N: \exp(X) \times [0, \text{dia}(X)] \rightarrow \exp(X)$ is a contraction, where $\text{dia}(X)$ is the diameter of X .

We shall assume henceforth that $\text{dia}(X) = 1$.

3. Citations and lemmas. We cite the first author's topological characterization of l^2 , and, because it is lengthy, we cite also Wojdysławski's topological characterization of AR's.

PROPOSITION 1. *Characterization of separable Hilbert space [11]. A complete separable metric AR Y is homeomorphic to l^2 if and only if each map $f: Q \times N \rightarrow Y$ may be approximated by closed embeddings.*

PROPOSITION 2. *Characterization of separable metric AR's [13]. A separable metric space Y is an AR if and only if it possesses a countable, dense subset $A = \{a_i\}_{i \in N}$ and two sequences $\mathcal{C} = \{C_j\}_{j \in N}$ and $\mathcal{D} = \{D_j\}_{j \in N}$ of subsets such that for some indexing $\mathcal{B} = \{B_j\}_{j \in N}$ of the finite subsets of A , the following conditions hold.*

- (1) $B_j \subset C_j$,
- (2) C_j may be deformed to a point in D_j .
- (3) If $B_i \subset B_j$, then $D_i \subset C_j$, and
- (4) if $\lim_{k \rightarrow \infty} B_{j_k} = y \in Y$, then $y = \lim_{k \rightarrow \infty} D_{j_k}$.

LEMMA 1. *The Hausdorff metric d for $\exp(X)$ is convex.*

PROOF. If $A, B \in \exp(X)$, then

$$C = \left\{ c \in X \mid \text{for some } a \in A \text{ and } b \in B \text{ with } d(a, b) = \min_{b' \in B} d(a, b') \text{ or} \right. \\ \left. d(a, b) = \min_{a' \in A} d(a', b), d(a, c) = d(b, c) = d(a, b)/2 \right\}$$

is a mid-point for A and B .

LEMMA 2. *X' and X^* are AR's.*

PROOF. This is essentially established by Wojdysławski in [13]; we sketch his argument here. Let $A = \{a_i\}_{i \in N}$ be a countable, dense subset of $\exp(X)$, and let $\mathcal{B} = \{B_j\}_{j \in N}$ be the collection of finite subsets of A indexed so that $i \neq j$ implies $B_i \not\subset B_j$. Let $1 = \varepsilon_1 < \varepsilon_2 < \dots < \varepsilon_i < \dots < 2$, and let $f: N \rightarrow N$ be a function such that $f(i) = 1$ if B_i is a singleton and if $B_i \subsetneq B_j$ then $f(i) < f(j)$.

Set

$$C_j = \{E \in \exp(X) \mid d(E, u(B_j)) < \varepsilon_{f(j)} \operatorname{dia}(B_j) \text{ and } E \text{ contains a member of } B_j\},$$

unless B_j is a single-point set, in which case $C_j = B_j$. Let D_j be the closure in $\exp(X)$ of C_j . Now (1), (3), and (4) of Proposition 2 hold. To contract C_j in D_j , define $F: C_j \times I \rightarrow D_j$ by setting $F(E, t) = \{x \in X \mid \text{for some } y \in E \text{ there are } x = x_0, x_1, \dots, x_n = y \text{ in } X \text{ such that (1) there are geodesics between } x_i \text{ and } x_{i+1} \text{ lying in } N_{\varepsilon_{f(j)} \operatorname{dia}(B_j)}(u(B_j)) \text{ and (2) } \sum_i d(x_i, x_{i+1}) \leq 3 \operatorname{dia}(B_j)t\}$. Then

$$F(E, 1) = u(D_j) = N_{\varepsilon_{f(j)} \operatorname{dia}(B_j)}(u(B_j)).$$

(Wojdysławski argues the continuity of F .)

To adapt this proof to our case is simple. For X' , we choose any countable, dense sequence $\mathcal{Q} = \{a_i\}_{i \in \mathbb{N}}$ in X' , and let $\mathfrak{B} = \{\mathfrak{B}_j\}_{j \in \mathbb{N}}$ be the finite subsets of $\mathcal{Q}$. For each i , let $k(i)$ be the least integer for which there is a representative A_i of a_i which is a member of $\exp^{k(i)}(X)$. This A_i is unique. Let $m(j) = \max\{k(i) \mid a_i \in \mathfrak{B}_j\}$ and proceed as above to define $\mathcal{C}_j$ and $\mathcal{D}_j$ as subsets of $\exp^{m(j)}(X)$ as well as the contraction of $\mathcal{C}_j$ in $\mathcal{D}_j$. This gives the necessary structure to X' . To extend to X^* is a triviality because X^* is the completion of X' so that X' is dense in X^* , and we may use exactly the same structure, checking only that if $\mathfrak{B}_{j_k} \rightarrow x \in X^*$, then $\mathcal{D}_{j_k} \rightarrow x$. This is true because if $d(\mathfrak{B}_j, x) < \varepsilon$, then $d(a_i, x) < \varepsilon$ for each $a_i \in \mathfrak{B}_j$, so that $\operatorname{dia}(\mathfrak{B}_j) < 2\varepsilon$ and $d(a_i, u(\mathfrak{B}_j)) < 2\varepsilon$. (We define $u(\mathfrak{B}_j) \in \exp^{m(j)}(X)$ by regarding $\mathfrak{B}_j$ as a finite subset of $\exp^{m(j)}(X)$.) Therefore, $\operatorname{dia}(\mathcal{D}_j) < 4\varepsilon$ ($\mathcal{D}_j$ is a subset of $\exp^{m(j)}(X)$ and the diameter is taken there), and so the distance (in X^*) from $\mathcal{D}_j$ to x is less than 5ε . Therefore, X^* is an AR.

LEMMA 3. *The maps $u_n: X^* \rightarrow \exp^n(X) \hookrightarrow X^*$ converge to the identity uniformly on compacta.*

PROOF. If $x \in X^*$ and $x' \in \exp^n(X)$ then $u_n(x') = x'$, so

$$d(u_n(x), x) \leq d(x, x') + d(u_n(x'), u_n(x)) \leq 2d(x, x').$$

Thus, $d(u_n(x), x) \leq 2d(x, \exp^n(X))$. Hence, for any $x, y \in X^*$ and $n \in \mathbb{N}$,

$$\begin{aligned} d(x, u_n(y)) &\leq d(x, u_n(x)) + d(u_n(x), u_n(y)) \\ &\leq 2d(x, \exp^n(X)) + d(x, y). \end{aligned}$$

If $K \subset X^*$ is compact, then for each $\varepsilon > 0$ there is a finite ε -net $k_1, \dots, k_m$ in K , and as $X' = \bigcup_i \exp^i(X)$ is dense in X^* , we may find an n such that if $j \geq n$, then $d(k_i, \exp^j(X)) < \varepsilon$. Therefore, for $x \in K$ and $j \geq n$,

$$\begin{aligned} d(u_j(x), x) &\leq d(u_j(x), u_j(k_i)) + d(u_j(k_i), k_i) + d(k_i, x) \\ &\leq 2d(x, k_i) + 2d(k_i, \exp^j(X)) < 4\varepsilon. \end{aligned}$$

LEMMA 4. *If $n > 0$, then $\exp^n(X)$ is a Hilbert cube. If $n \geq 0$ then $\exp^n(X)$ is a Z -set in $\exp^{n+m}(X)$, for all $m > 0$, in X' , and in X^* .*

PROOF. Since X is a nondegenerate Peano continuum, $\exp(X)$ is a Hilbert cube by [7]. For each n , the function $f_\varepsilon: \exp^{n+1}(X) \rightarrow \exp^{n+1}(X)$ defined by $f_\varepsilon(A) = \{N_\varepsilon(a) | a \in A\}$ takes values in the complement of $\exp^n(X)$ and is ε -close to the identity. It induces a map on X' and X^* with the same properties, so $\exp^n(X)$ is a Z -set in $\exp^{n+m}(X)$, X' , and X^* .

LEMMA 5. *If d' is any metric for X topologically equivalent to d , then the pair $(X_d^*, X_{d'}^*)$, is uniformly homeomorphic to (X_d^*, X_d^*) .*

PROOF. By compactness, the change of metric homeomorphism $(X, d') \rightarrow (X, d)$ which is the identity on the set X is a uniformly continuous homeomorphism with uniformly continuous inverse. The lemma follows from the functoriality of the construction of X^* and X' .

4. Proof of the main results.

THEOREM 1. *If X is a nondegenerate Peano continuum (with metric d) then X^* is homeomorphic with separable Hilbert space.*

PROOF. We have already noted (in Lemma 5), that because of the functoriality of the construction $X \rightarrow X^*$, the topological type of X^* is independent of the metric d , and we may therefore assume [5, 9] that it is convex and that X has diameter equal to one. By Lemma 2, X^* is an AR, so by Proposition 1 it suffices to show that any map $f: Q \times N \rightarrow X^*$ may be approximated by closed embeddings.

Let $\mathcal{U}$ be an open cover of X^* , and let $\alpha: X^* \rightarrow (0, 1]$ be a map such that for each $x \in X^*$, $\{y \in X^* | d(x, y) < \alpha(x)\}$ lies in some member of $\mathcal{U}$.

Now, by Lemma 3, we may choose a sequence $i(1) < i(2) < \dots$ of integers such that the map $f': Q \times N \rightarrow X'$ given by $f'(q, n) = u_{i(n)}(f(q, n))$ is within $\frac{1}{3}\alpha f(x)$ of $f(x)$. Using Lemma 4, we may assume that f' embeds each $Q \times \{n\}$ in $\exp^{i(n)}(X) \setminus \exp^{i(n)-1}(X)$.

Now define $g: Q \times N \rightarrow X' \subset X^*$ by

$$g(x) = \{\{f'(x)\}, N_{\alpha(f(x))/3}(f'(x))\} \in \exp^{i(n)+2}(X)$$

for $x \in Q \times \{n\}$. Then

$$d(f'(x), g(x)) \leq \alpha(f(x))/3$$

and

$$d(g(x), g(Q \times \{1, \dots, n-1\})) = \alpha(f(x))/3$$

(because $\text{dia } X = 1$ implies $\text{dia}(\exp^j X) = 1$ for all j). Also, g embeds each $Q \times \{n\}$ in $\exp^{i(n)+2}(X) \setminus \exp^{i(n)+1}(X)$.

To complete the proof, we need only to show that $\{g(Q_n)\}_n$ is discrete in X^* . To do this, suppose that for some sequence $\{x_j\}_{j=1}^\infty$ with $x_j \in Q \times n(j)$ for each j , and $n(j+1) > n(j)$, $g(x_j) \rightarrow x \in X^*$. Then $\alpha(f(x_j)) \rightarrow 0$, whence $f(x_j) \rightarrow x$ and $\alpha(x) = 0$, a contradiction.

Let R^N be the countable product of lines, and let $\Sigma \subset R^N$ be $\bigcup_{n \in N} \prod_{i \in N} [-n, n]_i$.

COROLLARY 1. *Let X be a nondegenerate Peano continuum (with metric d). There is a homeomorphism $f: X^* \rightarrow R^N$ such that $f(X') = \Sigma$.*

PROOF. By Theorem 1, X^* is homeomorphic to l^2 , which is homeomorphic to R^N [1]. R. D. Anderson [2] and Bessaga and Pełczyński [3] have characterized those subsets Y of R^N for which there is a homeomorphism $g: R^N \rightarrow R^N$ with $g(Y) = \Sigma$. (See [4] for a full treatment.) These sets are precisely those which may be expressed as countable unions $\bigcup_{n \in N} K_n$ of sequences of compacta having the property that for each compact set A of R^N , every open cover $\mathcal{U}$ of A in R^N , and each $m \in N$ there exists an $n \in N$ and an embedding $h: A \rightarrow K_n$ which is $\mathcal{U}$ -close to the inclusion $A \rightarrow R^N$ and which restricts to the identity on $A \cap K_m$. This is a topological property, and we need only verify that $X' = \bigcup_{n \in N} \exp^n(X)$ satisfies it in X^* . This is easily done. (We take $\exp^n(X) = K_n$.) Let $A, \mathcal{U}$, and m be as above. From the compactness of A there is an $\varepsilon > 0$ such that for each $a \in A$, $\{x \in X^* \mid d(x, a) < \varepsilon\}$ is contained in some element of $\mathcal{U}$. By Lemma 2, there is an $n > m$ such that u_n moves no point of A as much as $\varepsilon/2$. Since $u_n: X^* \rightarrow \exp^n(X)$ is a retraction, and $\exp^m(X)$ is a Z -set in the Hilbert cube $\exp^n(X)$ (Lemma 4), $u_n|_A$ may be approximated by an embedding $h: A \rightarrow \exp^n(X)$ which is the identity on $A \cap \exp^m(X)$ and is within $\varepsilon/2$ of $u_n|_A$. (See [6 or 4] for standard Z -set technique.) This is the desired embedding.

COROLLARY 2. *There is a homeomorphism $h: X^* \rightarrow l^2$ with $h(X') = l_\sigma^2$, where l_σ^2 is the linear span in l^2 of the standard Hilbert cube Q .*

PROOF. There is a homeomorphism $g: (R^N, \Sigma) \rightarrow (l^2, l_\sigma^2)$. ([4, Chapter I, Theorem 3.1(a)] states that Σ and l_σ^2 are homeomorphic, but the proof gives the stated fact.)

5. Conclusion: Problems. We have not given a natural construction from X of a compactification $\hat{X}$ of X^* into a Hilbert cube such that the pair $(\hat{X}, X^*)$ is homeomorphic to the pair (Q, P) , where $P = \{x \in l^2 \mid x = (t_1, t_2, \dots) \text{ and } |t_i| < 1/i\}$ is the pseudo-interior of Q . One should exist. Anderson [2] and Bessaga and Pełczyński [3, 4] have given topological characterizations of those subsets Y of Q which may be carried to P by homeomorphisms of Q ; moreover, the first author [10] has given a topological characterization of Q analogous to Proposition 1. We leave this as an open problem.

If α is a nontrivial action of a compact Lie group G on a Peano continuum X , there is, by functoriality, an induced action β on X' and on X^* . The fixed point set $F = \{x \in X^* \mid \beta(g, x) = x \text{ for each } g \in G\}$ of β is again homeomorphic to l^2 . (In fact, the pair (F, F') is homeomorphic to (l^2, l_σ^2) , where $F' = F \cap X'$.) Also, F is a Z -set in X^* . We leave open the problem of relating such actions defined directly on l^2 . Problem: Construct directly on l^2 a G -action which is topologically conjugate to β .

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