MONOTONIC SEMICONJUGACIES ONTO EXPANDING MAPS OF THE INTERVAL¹

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ABSTRACT. A contraction mapping is used to produce a semiconjugacy from a map τ_1 with maximum at c_1 to an expanding unimodal map with maximum at c_2 under the assumption that there is an interval J containing c_1 such that there is a one-to-one, order-preserving correspondence between the orbit of J under τ_1 and the orbit of c_2 under the expanding map.

Introduction. In [5] Milnor and Thurston prove the existence of semiconjugacies from a given piecewise monotonic map, τ_1 , with growth number $s > 1$, onto an (expanding) map of constant slope s . This semiconjugacy is exploited in [4] to get a “spectral decomposition” of the nonwandering set of such maps. In [1] it is shown that any piecewise monotonic map with unique itineraries can be used as a “model” for other piecewise monotonic maps with the same kneading sequence if the orbits of the various turning points are infinite and disjoint. All of the above rely heavily on the machinery of “kneading sequences” developed by Milnor and Thurston. In this note we consider expanding unimodal maps τ_2 . Using simple function space methods we show that such maps are “models” for various classes of maps τ_1 which possess a closed interval whose orbit is in a one-to-one, order-preserving correspondence with the orbit of the turning point of τ_2 .

Discussion. Consider a pair of maps $\tau_1, \tau_2: [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:

- (1) τ_1 is piecewise monotonic but not necessarily continuous; i.e. there exists a point $0 < c_1 < 1$ such that τ_1 is nondecreasing on $[0, c_1]$ and nonincreasing on $[c_1, 1]$.
- (2) τ_2 is unimodal, continuous, and expanding; i.e. τ_2 is strictly increasing on $[0, c_2]$ and strictly decreasing on $[c_2, 1]$ and in addition there exists a constant $\lambda > 1$ such that $|\tau_2(x) - \tau_2(y)| \geq \lambda |x - y|$, for x and y in either subinterval. Also, we shall always have $\tau_2(0) = \tau_2(1) = 0$.

THEOREM 1. *Suppose τ_1, τ_2 satisfy (1) and (2) above and in addition there exists a closed interval J (possibly degenerate) containing c_1 such that either*

- (3) *the intervals $\tau_1^n(J)$ are disjoint for $n = 0, 1, 2, \dots$ and the images $\tau_2^n(c_2)$ are all distinct for $n = 0, 1, \dots$ or*

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(3') the intervals $\tau_1^i(J)$ are disjoint for $i = 0, 1, \dots, n + k - 1$ but $\tau_1^{n+k}(J) \subset \tau_1^k(J)$. Similarly the points $\tau_2^i(c_2)$ are all distinct for $i < n + k$ but $\tau_2^{n+k}(c_2) = \tau_2^k(c_2)$.

In either case we also assume

$$(4) \tau_1^i(J) < \tau_1^j(J) \Leftrightarrow \tau_2^i(c_2) < \tau_2^j(c_2).$$

There then exists a unique monotonically increasing map $\psi: [0, 1] \rightarrow [0, 1]$ such that $\psi \circ \tau_1 = \tau_2 \circ \psi$ and $\psi([0, c_1]) \subseteq [0, c_2]$; $\psi([c_1, 1]) \subseteq [c_2, 1]$.

PROOF. Let Ψ be the set of nondecreasing functions $\psi: [0, 1] \rightarrow [0, 1]$ such that $\psi(\tau_1^i(J)) = \tau_2^i(c_2)$ for all i . Now Ψ is a complete metric space when it is given the uniform metric. It is clearly nonempty because of (4). For $\psi \in \Psi$ define $T(\psi)$ by

$$T\psi(x) = \begin{cases} \tau_{2,1}^{-1}(\psi(\tau_1(x))) & \text{for } x \in [0, c_1], \\ \tau_{2,2}^{-1}(\psi(\tau_1(x))) & \text{for } x \in [c_1, 1], \end{cases}$$

where $\tau_{2,1} = \tau_2|_{[0, c_2]}$ and $\tau_{2,2} = \tau_2|_{[c_2, 1]}$.

Now for any x , $\psi(\tau_1(x)) \leq \psi(\tau_1(c_1)) = \tau_2(c_2)$ since ψ is monotonic. Thus $\tau_{2,i}^{-1}$ ($i = 1, 2$) and therefore $T(\psi)(x)$ are defined. Moreover, $\tau_{2,i}^{-1}(\psi(\tau_1(c_1))) = \tau_{2,i}^{-1}(\tau_2(c_2)) = c_2$ for $i = 1, 2$. Thus $T\psi(c_1) = c_2$. To check that $T(\psi) \in \Psi$ notice that $T(\psi)$ is monotone by construction since τ_1 and τ_2 are piecewise monotone. To verify that $T(\psi)(\tau_1^i(J)) = \tau_2^i(c_2)$, suppose, for example, that $\tau_1^i(J) < c_1$. Then $T(\psi)(\tau_1^i(J)) = \tau_{2,1}^{-1}(\psi(\tau_1^{i+1}(J))) = \tau_{2,1}^{-1}(\tau_2^{i+1}(c_2))$. But $\tau_1^i(J) < c_1 \Leftrightarrow \tau_2^i(c_2) < c_2$ by (4). Thus $\tau_{2,1}^{-1}(\tau_2(\tau_2^i(c_2))) = \tau_2^i(c_2)$.

Moreover T is a contraction mapping since

$$|T\psi_1 - T\psi_2| = \sup_x |T\psi_1(x) - T\psi_2(x)|$$

and

$$\begin{aligned} \sup_{x \leq c_1} |T\psi_1(x) - T\psi_2(x)| &= \sup_{0 \leq x \leq c_1} |\tau_{2,1}^{-1}\psi_1\tau_1(x) - \tau_{2,1}^{-1}\psi_2\tau_1(x)| \\ &\leq \left(\sup_{0 \leq x \leq c_1} |\psi_1\tau_1(x) - \psi_2\tau_1(x)| \right) / \lambda \\ &\leq \left(\sup_y |\psi_1(y) - \psi_2(y)| \right) / \lambda \\ &= (|\psi_1 - \psi_2|) / \lambda. \end{aligned}$$

Similarly, $\sup_{c_1 \leq x \leq 1} |T\psi_1(x) - T\psi_2(x)| \leq (|\psi_1 - \psi_2|) / \lambda$.

Thus T has a unique fixed point ψ_0 . It is clear that ψ_0 satisfies $\tau_2 \circ \psi_0 = \psi_0 \circ \tau_1$ and $\psi_0([0, c_1]) \subseteq [0, c_2]$, $\psi_0([c_1, 1]) \subseteq [c_2, 1]$. It is straightforward to show that any map ψ satisfying the above conditions is a fixed point of T .

COROLLARY 1. If τ_1 is continuous then the semiconjugacy ψ_0 is also continuous.

PROOF. The set $\mathcal{C}$ of continuous monotone functions in Ψ is closed and invariant under T . Thus it suffices to show that $\mathcal{C}$ is nonempty. That is we must show that the correspondence $\tau_1^i(J) \mapsto \tau_2^i(c_2)$ extends to a continuous map on the closure of the

union of the $\tau_1^i(J)$. For then we can extend linearly on each complementary interval. Under condition (3') this is immediate. Under condition (3), it clearly suffices to show that for any point x such that

$$\tau_1^{n_k}(J) \uparrow x \quad \text{and} \quad \tau_1^{m_k}(J) \downarrow x,$$

the corresponding limits for the sequences $\{\tau_2^{n_k}(c_2)\}$ and $\{\tau_2^{m_k}(c_2)\}$ are equal. That is, if we set $A = \lim_{k \rightarrow \infty} \tau_2^{n_k}(c_2)$ and $B = \lim_{k \rightarrow \infty} \tau_2^{m_k}(c_2)$, then $A = B$.

Suppose $A < B$ and $I = [A, B]$. If $c_2 \notin \tau_2^n(I)$ for $n = 0, 1, 2, \dots$, then the length of $\tau_2^n(I)$ is greater or equal to λ^n times length of I (condition (2)). This is clearly impossible and so we may suppose that $c_2 \notin \tau_2^k(I)$, $k = 0, 1, \dots, n-1$, and $c_2 \in \tau_2^n(I)$. Therefore $\tau_2^n|I$ is a homeomorphism. Suppose for definiteness that it is increasing. Then $\tau_2^{n+n_k}(c_2)$ increases to $\tau_2^n(A)$ and $\tau_2^{n+m_k}(c_2)$ decreases to $\tau_2^n(B)$ (for large k) where $\tau_2^n(A) \leq c_2 \leq \tau_2^n(B)$. Therefore by (4) we have

$$\tau_1^{n+n_k}(J) \leq J \leq \tau_1^{n+m_k}(J) \quad (\text{for large } k),$$

but $\lim_{k \rightarrow \infty} \tau_1^{n+n_k}(J) = \tau_1^n(x) = \lim_{k \rightarrow \infty} \tau_1^{n+m_k}(J)$ and so $J = \tau_1^n(x)$. Thus $J = \{c_1\} = \{\tau_1^n(x)\}$.

Therefore we can reduce the question to a consideration of the case where $\tau_1^{p_k}(c_1) \uparrow c_1$, $\tau_1^{q_k}(c_1) \downarrow c_1$ and $\lim_{k \rightarrow \infty} \tau_2^{p_k}(c_2) = A$, $\lim_{k \rightarrow \infty} \tau_2^{q_k}(c_2) = B$ where $A \leq c_2 \leq B$ and we wish to show $A = B$. Assume $A < B$ and let $I = [A, c_2]$. As above we may assume that c_2 does not belong to the interior of $\tau_2^k(I)$, $k = 0, \dots, n-1$, and c_2 belongs to the interior of $\tau_2^n(I)$. Reasoning exactly as above, we conclude that $\tau_1^n(c_1) = c_1$. This contradicts (3') and completes the proof.

Note. If $\tau_1(0) = \tau_1(1) = 0$, then the fixed point ψ_0 must have $\psi_0(0) = 0$ and $\psi_0(1) = 1$. Thus in the above corollary ψ_0 is onto.

THEOREM 2. Suppose τ_1, τ_2 satisfy conditions (2), (3') and (4) of Theorem 1 but in place of piecewise monotonicity of τ_1 we assume

(1') τ_1 is continuous with $\tau_1(x) \leq \tau_1(c_1)$ for all x and if J is the closed interval of condition (4) we assume that if $\tau_1^i(J) < \tau_1^j(J)$ are adjacent (i.e. there is no k with $\tau_1^i(J) < \tau_1^k(J) < \tau_1^j(J)$), then τ_1 maps the closed interval between $\tau_1^i(J)$ and $\tau_1^j(J)$ onto the interval between $\tau_1^{i+1}(J)$ and $\tau_1^{j+1}(J)$.

Then the conclusion of Corollary 1 holds.

PROOF. The proof proceeds as in Theorem 1 except that we modify Ψ as follows: $\psi \in \Psi \Leftrightarrow \psi$ is continuous, $\psi(\tau_1^i(J)) = \tau_2^i(c_2)$, and ψ maps the closed interval bounded by the adjacent intervals $\tau_1^i(J) < \tau_1^j(J)$ onto $[\tau_2^i(c_2), \tau_2^j(c_2)]$. We need only check that $T(\Psi) \subseteq \Psi$. If $\psi \in \Psi$, $x \in [0, 1]$ then $\tau_1(x) \leq \tau_1(c_1)$ and so $\psi\tau_1(x) \leq \tau_2(c_2)$ by (1''). Thus $T\psi(x)$ is well defined. Finally $T(\psi)$ is continuous and if $\tau_1^i(J) < \tau_1^j(J) < c_1$ (for example) then $\tau_1^{i+1}(J) < \tau_1^{j+1}(J)$ thus

$$(T(\psi))([\tau_1^i(J), \tau_1^j(J)]) = \tau_2^{-1}([\tau_2^{i-1}(c_2), \tau_2^{j-1}(c_2)]) = [\tau_2^i(c_2), \tau_2^j(c_2)]$$

using conditions (4) and (1'). Thus $T(\psi) \in \Psi$ and the theorem follows.

EXAMPLE. Suppose $\tau_1: [0, 1] \rightarrow [0, 1]$ is any continuous, onto map with $\tau_1(0) = \tau_1(1) = 0$. If $\tau_1(c) = 1$, there exists a unique continuous semi-conjugacy ψ from τ_1 to the triangle map

$$\tau_2(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} \leq x \leq 1, \end{cases}$$

such that $\psi([0, c]) \subseteq [0, \frac{1}{2}]$ and $\psi([c, 1]) \subseteq [\frac{1}{2}, 1]$.

Note. The solution to the conjugacy equation need not be unique in general. Consider the Chebyshev polynomials $T_n(x) = \cos(n \cos^{-1} x)$ from the closed interval $[-1, 1]$ into itself. For example $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$. It is well known that T_2 is topologically conjugate to the triangle map $\tau(x) = 2|x| - 1$. Suppose ψ is the conjugacy i.e. $\psi \circ T_2 = \tau \circ \psi$. Since $T_2 \circ T_n = T_n \circ T_2$ for $n = 2, 3, \dots$, we have $(\psi \circ T_n) \circ T_2 = \psi \circ T_2 \circ T_n = \tau \circ (\psi \circ T_n)$ and $\psi \circ T_n$ also satisfies the conjugacy equation for $n = 2, 3, \dots$. Our uniqueness results are unaffected, however, since in Theorem 1 we required the semiconjugacy to be monotonic whereas in Theorem 2 we required the semiconjugacy to preserve the intervals about the turning point (here $x = 0$). However $\psi \circ T_n$ maps $[-1, 0]$ onto all of $[-1, 1]$.

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