CARDINALITIES OF FIRST COUNTABLE R -CLOSED SPACES

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ABSTRACT. It is now well known that first countable compact Hausdorff spaces are either countable or have cardinality c . The situation for first countable H -closed spaces is that they have cardinality less than or equal c , and it is at least consistent that they may have cardinality $\aleph_1 < c$. We show that the situation is quite different for first countable R -closed spaces. We begin by constructing an example which has cardinality $\aleph_1$. Let λ_0 be the smallest cardinal greater than c which is not a successor. For each cardinal κ with $c \leq \kappa \leq \lambda_0$ we construct a first countable R -closed space of cardinality κ . We also construct a first countable R -closed space of cardinality λ_0 . This seems to indicate that there is no reasonable upper bound to the cardinalities of R -closed spaces as a function of their character.

1. Introduction. A regular space X is R -closed if X is closed in every regular space containing X as a subspace, X is *minimal regular* if X has no coarser regular topology, and X is *strongly minimal regular* (SMR) if X has a base for the closed sets consisting of R -closed subsets. It is known that if X is SMR, then X is minimal regular and if X is minimal regular, then X is R -closed. A *regular filter* $\mathcal{F}$ on a space X is a filter with the property that for $A \in \mathcal{F}$, there is an open $U \in \mathcal{F}$ such that $\text{cl } U \subset A$. We shall assume that a regular filter consists of open sets. A filter $\mathcal{F}$ is *free* if $\bigcap \{\text{cl } A : A \in \mathcal{F}\} = \emptyset$. An equivalent characterization of R -closed spaces is that there are no free regular filters, cf. [BPS].

The class of R -closed spaces does not seem to be a very well behaved class. For instance, not every regular space can be densely embedded in an R -closed space [He], and the product of R -closed spaces need not be R -closed [P]. These results are in sharp contrast to the classes of H -closed spaces and compact Hausdorff spaces, both of which are productive [BPS], and every Hausdorff (Tychonoff) space can be densely embedded in an H -closed (compact Hausdorff) space. One of the few positive results about R -closed spaces is in [DP1] where we show that every regular space can be embedded as a closed subspace of an R -closed space. This paper exhibits further distinctions between the class of R -closed spaces and the classes of compact Hausdorff spaces and H -closed spaces.

In the second section we construct an example of a first countable SMR space of cardinality $\aleph_1$. A space is $\aleph_0$ -bounded if every countable set has compact closure. Hechler [H] has given consistent examples of first countable separable SMR spaces of cardinality $\aleph_1$. However, our example requires no special set theoretic assumptions and our example is $\aleph_0$ -bounded. In the final section we recursively construct

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first countable SMR spaces for each cardinal κ with $c \leq \kappa \leq \lambda_0$ and a first countable SMR space of cardinality λ_0^ω .

A construction of R -closed spaces that has been put to good use by many authors is a technique due to Jones. In [DP1] we investigate this technique in some detail, but we will need a modification of it due to Stephenson [S]. We thank the referee for his or her useful suggestions.

1.1. Suppose that X is a regular space containing pairwise disjoint closed sets H_1 , H_2 and K . Suppose further that we have homeomorphisms $f_i: H_i \rightarrow K$. Let E be the equivalence relation on $X \times \mathbf{Z}$, where $\mathbf{Z}$ is the integers, defined by the rule $(x, i) E (y, j)$ if: (i) $x = y$ and $i = j$; (ii) $x = f_2(y)$ and $i + 1 = j$, (iii) $y = f_2(x)$ and $j + 1 = i$; (iv) $x = f_1(y)$ and $j + 1 = i$; (v) $y = f_1(x)$ and $i + 1 = j$; (vi) $f_1(x) = f_2(y)$ and $i + 2 = j$; or (vii) $f_1(y) = f_2(x)$ and $j + 2 = i$. Choose two new points $\{P_+, P_-\}$ and let $DJ(X) = \{P_+, P_-\} \cup (X \times \mathbf{Z})/E$ and $J(X) = \{P_+\} \cup (X \times N)/E|_{(X \times N)}$, where N is the positive integers. For each $n \in N$ let $V_n = \{P_+\} \cup X \times \{m \in \mathbf{Z}: m \geq n\}$ and $W_n = \{P_-\} \cup X \times \{m \in \mathbf{Z}: m \leq -n\}$. We topologize $DJ(X)$ by giving $(X \times \mathbf{Z})/E$ the quotient topology and letting $\{V_n: n \in N\}$ and $\{W_n: n \in N\}$ form neighborhood bases for P_+ and P_- respectively. With this topology $DJ(X)$ is a regular space, and we let $J(X)$ have the subspace topology.

PROPOSITION 1.2. *Let X be locally compact and Hausdorff, and let H_1, H_2, K, f_1 and f_2 be as in 1.1. Suppose that for some $n \in \omega$ and any sequence $U_0, \dots, U_{n+1}$ with $\text{cl } U_{i+1} \subset \text{int } \text{cl } U_i$ for each $i \leq n$, $\text{cl } U_{n+1}$ not compact implies that $\text{cl } U_0 \cap K$ is not compact. Then $DJ(X)$ is SMR and $J(X)$ is R -closed.*

PROOF. Since X is locally compact, so is $DJ(X) \setminus \{P_+, P_-\}$. By 1.2 of [DP1] and the fact that for each $m \in N$, V_m and W_m are homeomorphic to $J(X)$, it suffices to show that $J(X)$ is R -closed.

Suppose that $\mathcal{F}$ is a free regular filter on $J(X)$. Since $\mathcal{F}$ is free, there is a $U_0 \in \mathcal{F}$ such that $P_+ \notin \text{cl } U_0$. Suppose that $k \in N$ is such that $V_k \cap \text{cl } U_0 = \emptyset$. Let $m = k \cdot (n + 1)$ and recursively choose $\{U_i: i \leq m\} \subseteq \mathcal{F}$ such that $\text{cl } U_{i+1} \subset U_i$ for $i = 0, 1, \dots, m - 1$. Since $\mathcal{F}$ is free, for some $i < m$, $\text{cl } U_m \cap X \times \{i\}$ is not compact. Therefore, by assumption on X , $\text{cl } U_{m-(n+1)} \cap K \times \{i\}$ is not compact. Since $K \times \{i\}$ is identified with $H_2 \times \{i + 1\}$ by f_2 , we have that $\text{cl } U_{m-(n+1)} \cap X \times \{i + 1\}$ is not compact. By induction, this implies that $\text{cl } U_{m-m} \cap X \times \{k\}$ is not compact; however, this contradicts that $\text{cl } U_0 \cap V_k = \emptyset$.

2. The $\aleph_1$ example. Let us begin by establishing some notation. Let $\Lambda = \{\lambda_\alpha: \alpha < \omega_1\}$ be an order preserving listing of the limit ordinals less than ω_1 . For each $n \in \omega$, let $A_n = [(\omega_1 \setminus \Lambda) \times (n + 1) \cup \Lambda] \times \{n\}$, and let $X = \bigcup_{n \in \omega} A_n$. Recall that an ordinal is the set of its predecessors. For each $\alpha \in \omega_1$, let

$$X_\alpha = \bigcup_{n \in \omega} [\lambda_\alpha \setminus \Lambda \times (n + 1) \cup \{\lambda_\beta: \beta \leq \alpha\}] \times \{n\}.$$

We will construct, for $\alpha \in \omega_1$, a topology τ_α on X_α which is compact Hausdorff and has a basis of compact open sets. This construction is a modification of Vaughan's [V] construction of a first countable, countably compact, nonnormal

space and is indeed itself such a space. We have not been able to show that Vaughan's space satisfies the hypothesis of 1.2 and suspect it does not as it contains countably many pairwise disjoint clopen copies of ω_1 whose union is dense.

To help clarify the construction we give an informal description first. Each column A_n is $\omega_1 \times (n + 1)$ with the limit ordinals identified, and will be homeomorphic to ω_1 . We think of each column $A_n \setminus (\Lambda \times \{n\})$ as containing $n + 1$ columns. If we take a horizontal cross section at level $\alpha \in \omega_1 \setminus \Lambda$ by choosing the element out of the k th column of A_n for each $n > k$ this sequence will converge to the point in the k th column of A_k . The limit levels will compactify what is below them with, in addition, $\{(\lambda, n): 1 \leq n < \omega\}$ converging to $(\lambda, 0)$.

Suppose that $\alpha \in \omega_1$ and for $\beta < \alpha$ we have constructed a Hausdorff topology τ_β on X_β such that

- (a) τ_β has a countable base of compact open sets;
- (b) for each $n \in \omega$, $\gamma \in \lambda_\beta \setminus \Lambda$ and $k < n$ the point (γ, k, n) is isolated;
- (c) for each $k \in \omega$ and $\gamma \in \lambda_\beta \setminus \Lambda$, a neighborhood base for (γ, k, k) is $\{(\gamma, k, k)\} \cup \{(\gamma, k, m): m > n\}: k \leq n < \omega\}$;
- (d) for each $\delta \leq \beta$ and $n < \omega$ the set $\{(\gamma, k, n): \gamma < \lambda_\delta, k \leq n\} \cup \{(\lambda_\delta + 1 \cap \Lambda) \times \{n\}\}$ is compact;
- (e) for each $\delta \leq \beta$, $\{(\lambda_\delta, n): 0 < n < \omega\}$ converges to $(\lambda_\delta, 0)$; and
- (f) (X_δ, τ_δ) is an open subspace of (X_β, τ_β) for each $\delta < \beta$.

We will define τ_α by defining neighborhood bases at each of the points of $X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta$.

Step 1. For each $\gamma \in \lambda_\alpha \setminus \bigcup_{\beta < \alpha} \lambda_\beta + 1$ and $k < n < \omega$, we put $\{(\gamma, k, n)\}$ in τ_α .

Step 2. For each $\gamma \in \lambda_\alpha \setminus \bigcup_{\beta < \alpha} \lambda_\beta + 1$ and $k < \omega$, a neighborhood base for (γ, k, k) will be

$$\{(\gamma, k, k)\} \cup \{(\gamma, k, m): m > n\}: k \leq n < \omega\}.$$

Step 3. Let $\{\gamma_i: i \in \omega\} \subset \lambda_\alpha \setminus \Lambda$ be a strictly increasing sequence converging to λ_α . For each $0 < n < \omega$, let $B_0(n) = [(\gamma_0 \setminus \Lambda) \times (n + 1) \cup (\Lambda \cap \gamma_0)] \times \{n\}$ and recursively define, for $1 \leq i < \omega$,

$$B_i(n) = [(\gamma_i \setminus \Lambda) \times (n + 1) \cup (\Lambda \cap \gamma_i)] \times \{n\} \setminus B_{i-1}(n).$$

It is easily seen that for each $n \in \omega \setminus \{0\}$, $\{B_i(n): i \in \omega\}$ is locally finite in $X_\alpha \setminus (\{\lambda_\alpha\} \times \omega)$. In addition, by induction, assumptions (c), (d), and (f), each $B_i(n)$ is a compact set. Each point of $B_i(n)$ has a neighborhood base of compact open sets already defined. By the definition of these neighborhood bases we may choose, for each $n > 0$, a locally finite collection of compact open subsets, $\{C_i(n): i \in \omega\}$, of $X_\alpha \setminus (\{\lambda_\alpha\} \times \omega)$ so that $C_i(n) \cap A_n = B_i(n)$ and $C_i(n) \cap A_j = \emptyset$ for $j \leq \max\{i, n\}$, $j \neq n$. Recursively define for $1 \leq n < \omega$ a neighborhood base at (λ_α, n) as follows: For each $j < \omega$,

$$W_j((\lambda_\alpha, n)) = [\{(\lambda_\alpha, n)\} \cup \{C_i(n): j < i < \omega\}] \setminus \bigcup \{W_n((\lambda_\alpha, m)): 0 < m < n\}.$$

Note that for $m < n$, $W_n((\lambda_\alpha, m)) \cap A_n = \emptyset$ and that it can be shown by induction on n that $W_i((\lambda_\alpha, n))$ is compact open. To check (d), observe that for any $j \in \omega$, $(A_n \cap X_\alpha) \setminus W_j((\lambda_\alpha, n)) = \bigcup \{B_i(n): i \leq j\}$ which is compact.

Step 4. By construction $X_\alpha \setminus \{(\lambda_\alpha, 0)\}$ has a countable base of compact open sets. We let X be the one point compactification of $X_\alpha \setminus \{(\lambda_\alpha, 0)\}$. This clearly satisfies conditions (e) and (f) and the others obviously hold by construction.

We let τ be the topology on X generated by the open base $\bigcup_{\alpha < \omega_1} \tau_\alpha$. With this topology X is first countable, zero dimensional (has a base of clopen sets), locally compact, $\aleph_0$ -bounded, nonnormal and has cardinality $\aleph_1$. To see that X is $\aleph_0$ -bounded we simply note that any countable set is contained in some X_α . Nonnormality will be a trivial consequence of the following.

Let U be an open set containing a closed noncompact subset of A_n , where $n \in \omega$. We will show that $\text{cl } U \cap A_0$ is not compact. Since A_n is homeomorphic to ω_1 , $A_n \setminus U$ is countable [GJ]. Therefore there is an $\alpha < \omega_1$ such that for $\gamma > \alpha$ and $\gamma \notin \Lambda$, $(\gamma, n, n) \in U$. For each such γ , there is a $k \in \omega$ such that $\{(\gamma, n, j) : j > k\} \subset U$ since U is open. It follows that there are an uncountable set $\Gamma \subset \omega_1 \setminus \Lambda$ and a $k \in \omega$ such that for each $\gamma \in \Gamma$, $\{(\gamma, n, j) : j > k\} \subset U$. Now, for each limit λ of Γ and $j > k$, $(\lambda, j) \in \text{cl } U$ since $(\lambda, j) \in \text{cl}\{(\gamma, n, j) : \gamma \in \Gamma\}$. Therefore $(\lambda, 0) \in \text{cl } U$ for each limit λ of Γ . Since Γ is uncountable, $\text{cl } U \cap A_0$ is not compact.

Now since each A_n is homeomorphic to ω_1 if we let $H_1 = A_1$, $H_2 = A_2$ and $K = A_0$, we have that X satisfies the conditions of 1.2 with $n = 0$. We have shown

THEOREM 2.1. *$DJ(X)$ is a first countable SMR space with cardinality $\aleph_1$.*

3. Cardinals greater than c . In this section we will construct first countable SMR spaces with cardinalities up to λ_0 where λ_0 is the first singular cardinal greater than c . Assuming GCH, $\lambda_0 = \aleph_\omega$. We will make extensive use of a space developed by Mrowka and Isbell then modified by Stephenson [S] and Hechler [H].

3.1. Suppose D is an infinite discrete set and R is a family of infinite pairwise almost disjoint countable subsets of D . (A and B are *almost disjoint* if $|A \cap B| < \omega$.) We will give $X = D \cup R$ the following topology. D is open and discrete while a neighborhood base for $r \in R$ is $\{\{r\} \cup r \setminus F : F \text{ is a finite subset of } r\}$. It is easily shown that X is a first countable locally compact Hausdorff space.

Suppose that R contains disjoint subsets H, K with $H \cup K = R$ and $|H| = |K|$. Suppose further that there is an $n \in \omega$ such that for any sequence $U_0, \dots, U_{n+1} \subset D$ with $\text{cl } U_{i+1} \subset \text{int } \text{cl } U_i$ for $i \leq n$, $|\text{cl } U_{n+1} \cap H| \geq \aleph_0$ implies $|\text{cl } U_0 \cap K| > \aleph_0$. It is then clear that if we let $H_1 \cup H_2 = H$ with $|H_1| = |H_2|$ and f_i be any isomorphism from H_i to K for $i = 1, 2$, then X satisfies the conditions 1.2. Therefore $DJ(X)$ would be SMR. This will be our plan, to recursively construct X_κ , for $c \leq \kappa \leq \lambda_0$, so that $X_\kappa = D_\kappa \cup R_\kappa$ with $|R_\kappa| = \kappa$ and R_κ satisfying the above.

To begin we borrow a result by Hechler [H] whose purpose was to remove the assumption of the continuum hypothesis from an example by Stephenson [S].

3.2. THEOREM. *There is a maximal almost disjoint family R_c of infinite subsets of ω such that $|R_c| = c$ and there is a subset $H_c \subset R_c$ with $|H_c| = c$ such that for $U \subset \omega$, $|\{r \in H_c : |U \cap r| = \aleph_0\}| \geq \aleph_0$ implies $|\{r \in R_c \setminus H_c : |U \cap r| = \aleph_0\}| = c$.*

For a space $X = D \cup R$ as in 3.1 we will say that (X, H, K) satisfies $(*)_n$ if the following hold: (i) $H \subset R$ and $K = R \setminus H$; (ii) $|H| = |K| = |R|$; and (iii) for

$U_0, \dots, U_{n+1} \subseteq D$ with $\text{cl } U_{i+1} \subset \text{int cl } U_i$ for $i \leq n$, then $|\text{cl } U_{n+1} \cap H| \geq \aleph_0$ implies $|\text{cl } U_0 \cap K| > \aleph_0$. Let $D_c = \omega$ and let R_c and H_c be as in 3.2. Then $X_c = D_c \cup R_c$ with the topology as in 3.1 satisfies $(*)_0$. Indeed, for $r \in R_c$ and $W \subset D_c$, $r \in \text{cl } W$ iff $|r \cap W| = \aleph_0$. Let $K_c = R_c/H_c$.

Therefore if $U_0, U_1 \subset D_c$ are as in (iii) with $|\text{cl } U_1 \cap H_c| \geq \aleph_0$ then by 3.2 $|\text{cl } U_1 \cap K_c| = c$. Hence $|\text{cl } U_0 \cap K_c| > \aleph_0$.

3.3. *The main construction.* Let κ be a cardinal such that $\kappa^\omega = \kappa$. Suppose that $X_\kappa = D_\kappa \cup R_\kappa$ is as in 3.1 with $|D_\kappa| \leq |R_\kappa| = \kappa$. Suppose also that $n \in \omega$ and that $(X_\kappa, H_\kappa, K_\kappa)$ satisfy $(*)_n$. We shall let $D_{\kappa^+} = D_\kappa \times \kappa^+$ and define an almost disjoint family, R_{κ^+} , of countable subsets of D_{κ^+} and subsets H_{κ^+} and K_{κ^+} of R_{κ^+} so that $(X_{\kappa^+}, H_{\kappa^+}, K_{\kappa^+})$ satisfies $(*)_{n+1}$.

An ordinal δ is said to have *countable cofinality*, $\text{cf } \delta = \omega$, if there is a sequence $\{\gamma_n: n \in \omega\} \subseteq \delta$ such that $\delta = \sup\{\gamma_n\}$. For each $\delta < \kappa^+$ with $\text{cf } \delta = \omega$ we shall construct a set $R(\delta)$ as follows. Choose any strictly increasing sequence $\{\gamma_n: n \in \omega\}$ converging to δ . Let $R(\delta)$ be a maximal family of almost disjoint infinite countable subsets of $D_\kappa \times \delta$ such that for each $r \in R(\delta)$ and $n \in \omega$, $|r \cap D_\kappa \times \gamma_n| < \aleph_0$. Note that $|R(\delta)| \leq |D_\kappa \times \delta|^\omega \leq \kappa^\omega = \kappa$. Fix an injection f_δ from $R(\delta)$ into H_κ . We define $R_{\kappa^+} = \{r \cup (f_\delta(r) \times \{\delta\}): \delta < \kappa^+ \text{ with } \text{cf } \delta = \omega \text{ and } r \in R(\delta)\} \cup \{r \times \{\gamma\}: \gamma < \kappa^+, r \in R_\kappa, \text{ and if } \text{cf } \gamma = \omega, r \notin f_\gamma(R(\gamma))\}$. Each element of R_{κ^+} is a countable subset of D_{κ^+} and R_{κ^+} is an almost disjoint family. Let us define $H_{\kappa^+} = \{r \in R_{\kappa^+}: \text{there are an } r_1 \in H_\kappa \text{ and } \gamma < \kappa^+ \text{ with } r_1 \times \{\gamma\} \subset r\}$. Also let $K_{\kappa^+} = R_{\kappa^+} \setminus H_{\kappa^+} = \{r \in R_{\kappa^+}: \text{there are an } r_1 \in K_\kappa \text{ and } \gamma < \kappa^+ \text{ with } r_1 \times \{\gamma\} = r\}$. Note that $|H_{\kappa^+}| = |H_\kappa| \cdot \kappa^+ = |K_\kappa| \cdot \kappa^+ = |K_{\kappa^+}|$. Let us check that $(X_{\kappa^+}, H_{\kappa^+}, K_{\kappa^+})$ satisfies $(*)_{n+1}$. Let $U_0, \dots, U_{n+2} \subset D_{\kappa^+}$ with $\text{cl } U_{i+1} \subset \text{int cl } U_i$ for $i \leq n+1$ and $|\text{cl } U_{n+2} \cap H_{\kappa^+}| \geq \aleph_0$. Suppose first that there is a $\gamma < \kappa^+$ such that $\text{cl } U_{n+2} \cap \{r \in H_{\kappa^+}: \text{there is an } r_1 \in H_\kappa \text{ with } r_1 \times \{\gamma\} \subset r\}$ is infinite. Therefore, since $(X_\kappa, H_\kappa, K_\kappa)$ satisfies $(*)_n$, $\text{cl } U_0 \cap \{r \in K_{\kappa^+}: \text{there is an } r_1 \in K_\kappa \text{ with } r_1 \times \{\gamma\} = r\}$ is uncountable. Hence $|\text{cl } U_0 \cap K_{\kappa^+}| > \aleph_0$. On the other hand, suppose that there is an infinite increasing sequence $\{\gamma_j: j \in \omega\} \subset \kappa^+$ such that, for each $j \in \omega$, there are an $r_1(j) \in H_\kappa$ and an $r(j) \in H_{\kappa^+} \cap \text{cl } U_{n+2}$ with $r_1(j) \times \{\gamma_j\} \subset r(j)$. Therefore, for each $j \in \omega$, $U_{n+2} \cap D_\kappa \times (\gamma_j + 1)$ is infinite. So let $\delta = \sup\{\gamma_j: j \in \omega\}$ and choose an infinite set $S \subset R(\delta)$ such that $s \cap U_{n+2}$ is infinite for each $s \in S$. This can be done since $s \cap D_\kappa \times \gamma_j$ is finite for each $j \in \omega$ and $s \in R(\delta)$. Therefore, $s \cup (f_\delta(s) \times \{\delta\}) \in \text{cl } U_{n+2}$ for each $s \in S$ and by the above $|\text{cl } U_0 \cap K_{\kappa^+}| > \aleph_0$.

We now obtain the following results.

THEOREM 3.4. *For each cardinal κ with $c \leq \kappa < \lambda_0$ there is a first countable SMR space with cardinality κ .*

PROOF. Observe that for a cardinal κ with $\text{cf}(\kappa) > \omega$, $\kappa^\omega = \sum\{\alpha^\omega: \alpha < \kappa\}$. It follows that $(c^+)^{\omega} = \sum\{|\alpha^\omega|: \alpha < c^+\} = c^+ \cdot c = c^+$. Similarly, by induction, $\kappa^\omega = \kappa$ for $c \leq \kappa < \lambda_0$. Therefore, by 3.2 and 3.3, we have a space X_κ of cardinality κ such that $(X_\kappa, H_\kappa, K_\kappa)$ satisfies $(*)_n$ for some $n \in \omega$. We partition H_κ into two sets H_1 and H_2 with $|H_1| = |H_2|$. By 1.2, $DJ(X_\kappa)$ is a first countable SMR space.

The following result follows easily from the results in [P].

THEOREM 3.5. *A product of first countable SMR spaces is itself SMR.*

COROLLARY 3.6. *There are first countable, SMR spaces of cardinality λ_0 and λ_0^ω , respectively.*

PROOF. For $n \in \omega$, let κ_n be the n th successor of c . Thus, $\lambda_0 = \Sigma\{\kappa_n: n \in \omega\}$. For each $n \in \omega$, let X_n be a first countable, SMR space of cardinality κ_n . Let Y be the topological sum of $\{X_n: n \in \omega\}$ (we write $Y = \bigcup \{X_n: n \in \omega\}$) and $X = \{P_+\} \cup Y$. Now, $U \subseteq X$ is defined to be open if $U \cap Y$ is open in Y and if $P_+ \in U$, there is $m \in \omega$ such that $U \supseteq \bigcup \{X_n: n \geq m\}$. It is immediate that X is first countable and R -closed and has cardinality λ_0 . Since each X_n is SMR, then X is also SMR. Finally let $Z = \Pi\{X_n: n \in \omega\}$. By 3.5, Z is SMR and is clearly first countable. The cardinality of Z is $|\Pi\{X_n: n \in \omega\}| = \lambda_0^\omega$.

REFERENCES

- [BPS] M. P. Berri, J. R. Porter and R. M. Stephenson, Jr., *A survey of minimal topological spaces*, General Topology and its Relations to Modern Analysis and Algebra, III (Proc. Conf., Kanpur, 1968), Academia, Prague, 1971, pp. 93–114.
- [DP1] A. Dow and J. Porter, *Embedding in R -closed spaces*, Topology Appl. **15** (1983), 45–58.
- [DP2] ———, *Cardinalities of H -closed spaces*, Topology Proc. **7** (1982).
- [G] A. A. Gryzlow, *Two theorems on the cardinality of topological spaces*, Soviet Math. Dokl. **21** (1980), 506–509.
- [H] S. H. Hechler, *Two R -closed spaces revisited*, Proc. Amer. Math. Soc. **56** (1976), 303–309.
- [He] H. Herrlich, *T_v -Abgeschlossenheit und T_v -Minimalität*, Math. Z. **88** (1965), 285–294.
- [P] D. Pettey, *Products of regular-closed spaces*, Topology Appl. **14** (1982), 189–199.
- [S] R. M. Stephenson, Jr., *Two R -closed spaces*, Canad. J. Math. **24** (1972), 286–292.
- [V] J. E. Vaughan, *A countably compact, first countable, non-normal T_2 -space*, Proc. Amer. Math. Soc. **15** (1979), 339–342.

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