EMBEDDINGS IN $G(1, 3)$

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ABSTRACT. We give a description of nonsingular surfaces of degree $d \leq 8$ in the Grassmannian $G(1, 3)$ embedded in $\mathbf{P}^5$.

1. Preliminaries. Let $G = G(1, 3)$ be the Grassmann variety of projective lines in a fixed $\mathbf{P}^3$. Fix an embedding of G in $\mathbf{P}^5$. Let $Y \subset G$ be a nonsingular surface. This embedding is uniquely determined by a short exact sequence of vector bundles on Y ,

$$(1) \quad 0 \rightarrow E \rightarrow \mathcal{O}_Y^4 \rightarrow Q \rightarrow 0,$$

where $\text{rank } Q = \text{rank } E = 2$ and $\wedge^2 Q$ is the very ample line bundle embedding Y in $\mathbf{P}^5$. In the Chow ring $A(G)$ of G , $Y \sim aZ_1 + bZ_2$, where the Z_i are the two Schubert cycles of dimension 2 in $A(G)$, $a = c_2(Q)$, $b = c_2(\check{E})$, and the degree of Y in $\mathbf{P}^5$ is $d = c_1^2(Q) = c_1^2(\check{E}) = a + b$. If $Y \neq Z_i$, then $a, b > 0$. We denote by (a, b) the class of Y in $A(G)$. For background see [9]. Throughout we work over $\mathbf{C}$.

PROPOSITION 1.1. *With Y and (a, b) as above, let H be a hyperplane section on Y , K the canonical divisor on Y , and $N_{Y/G}$ the normal bundle on Y in G . Then*

$$(2) \quad a^2 + b^2 = c_2(N_{Y/G}) = 7d + 4KH + 2K^2 - 12 - 12p_a(Y).$$

PROOF. By [9, Appendix] the tangent bundle of G restricted on Y is $T_{GY} = Q \otimes \check{E}$. Hence, tensoring (1) by $\check{E}$ and calculating total Chern classes, we get

$$c_1(T_{GY}) = 4c_1(\check{E}) = 4H \quad \text{and} \quad c_2(T_{GY}) = 7c_1^2(\check{E}) = 7d.$$

Hence, from the short exact sequence $0 \rightarrow T_Y \rightarrow T_{GY} \rightarrow N_{Y/G} \rightarrow 0$, we get

$$c_1(N_{Y/G}) = 4H + K \quad \text{and} \quad c_2(N_{Y/G}) = 7d - c_2(T_Y) + 4KH + K^2.$$

(2) follows by Riemann-Roch. Q.E.D.

PROPOSITION 1.2. *With Y and (a, b) as above, let C be a nonsingular curve on Y representing the hyperplane section H . Assume Y lies in $\mathbf{P}^4$. Then:*

- (i) *The genus of C is $g = ab - (a + b) + 1$.*
- (ii) *If $d = 2n$, then $(a, b) = (n, n)$ and Y is a complete intersection, $Y = G \cap \mathbf{P}^4 \cap S_n$, where S_n is a hypersurface in $\mathbf{P}^5$ of degree n .*

PROOF. $Y \subset \mathbf{P}^4$ and, hence, from [5, p. 434] we have

$$(3) \quad d^2 - 10d - 5HK - 2K^2 + 12 + 12p_a(Y) = 0.$$

Received by the editors June 10, 1982 and, in revised form, April 12, 1983.

1980 *Mathematics Subject Classification.* Primary 14M15; Secondary 14J99, 14E25.

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0002-9939/83 \$1.00 + \$.25 per page

From (2) we have $d^2 - (a^2 + b^2) - 3d - HK = 0$, and (i) follows from the adjunction formula.

For (ii) we can assume Y does not lie in $\mathbf{P}^3$. (Indeed, if $Y \subset \mathbf{P}^3$, then Y is a nonsingular quadric, $a = b = 1$, and $Y = G \cap \mathbf{P}^3$.) $Y \subset Z = G \cap \mathbf{P}^4$. If Z is nonsingular, then $\text{Pic}(Z) \cong \text{Pic}(G) \cong \mathbf{Z}$, and Y is a complete intersection $Z \cap S_n \cap \mathbf{P}^4$. Hence, C is a complete intersection and (ii) follows from [6, Theorem 1]. If Z is singular, then since $d = 2n$, (ii) follows by [5, p. 352]. Q.E.D.

For a nonsingular surface Y in G ruled over a nonsingular curve T , we fix the following notation: $\pi: Y \rightarrow T$ is the natural projection, $\varphi: Y \rightarrow Y_0$ is a birational morphism where Y_0 is geometrically ruled over T , $\varphi(C) = X = mT_0 + nf$ in Y_0 , where C is a nonsingular curve representing a hyperplane section on Y , $e = -(T_0 \cdot T_0)_{Y_0}$.

Y_0 is obtained from Y by blowing down r exceptional divisors, and r_p is the multiplicity of a point $p \in X$ where $p = \varphi(E)$ where E is an exceptional divisor in Y . We have

$$(4) \quad \begin{aligned} X^2 &= m(2n - me) = C^2 + \sum_p r_p^2, \\ 2g(X) - 2 &= X^2 + me - 2n - m(2p_a(Y_0) + 2), \\ g(X) &= g(C) + \frac{1}{2} \sum_p r_p(r_p - 1), \quad K_{Y_0}^2 = K_Y^2 + r. \end{aligned}$$

In the next section we will also use the following

PROPOSITION 1.3 [4, Propositions 3.1, 3.2]. *Let Y be a rational ruled surface and C a nonsingular curve on Y . Then there exists a birational morphism $\varphi: Y \rightarrow F_e$, for some e , with the following properties:*

- (a) every multiple point of $\varphi(C) = X$ has multiplicity $r_p \leq m/2$;
- (b) if $e = 0$, then $m \leq n$,
- (c) if $e = 1$, then $r_p \leq n$;
- (d) if $m \geq 4$ and either $e \neq 1$, or $e = 1$ and $n \geq m/2$, then $C^2 \leq \frac{2m}{m-2}(g(C) - 1)$,
- (e) if $m \geq 4$, $e = 1$ and $n < m/2$, then

$$C^2 < \frac{2k}{k-3}(g(C) - 1), \quad \text{where } k = n + m.$$

PROPOSITION 1.4 [1, p. 115]. *Let Y be a surface in $\mathbf{P}^n$ not contained in a hyperplane. Let $d = \deg(Y)$. Then:*

- (i) If Y is not ruled, then $d \geq 2n - 2$, with equality holding only if $K = 0$.
- (ii) If Y is ruled, then $0 \geq p_a(Y) \geq n - d - 1$.

2. Surfaces of degree $d \leq 8$.

LEMMA 2.1. *Let H be a very ample divisor on a nonsingular curve C that embeds C in $\mathbf{P}^n$ so that C does not lie in $\mathbf{P}^{n-1}$. Let d be the degree of C . If $d \geq 2g + 1$, then $g = d - n$.*

PROOF. H is a nonspecial divisor on C . Hence $l(H) = d - g + 1$. By [7, p. 55] the embedding of C by H is projectively normal. Thus $l(H) = N + 1$. Q.E.D.

LEMMA 2.2. Let Y be a nonsingular surface in $\mathbf{P}^n$ that does not lie in a $\mathbf{P}^{n-1}$. Assume $\deg(Y) \geq 2g_C + 1$, where g_C is the genus of the nonsingular curve C given by a hyperplane section. Then $H^1(O_Y(m)) = 0$ for all m .

PROOF. $C \subset \mathbf{P}^{n-1}$ is projectively normal. Hence, from the commutative diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ H^0(O_{\mathbf{P}^{n-1}}(m)) & \rightarrow & H^0(O_C(m)) & \rightarrow & 0 \\ & \uparrow & \uparrow \varphi & & \\ H^0(O_{\mathbf{P}^n}(m)) & \rightarrow & H^0(O_Y(m)) & & \end{array}$$

we get that φ is surjective for all m . Hence $0 \rightarrow H^1(O_Y(m-1)) \rightarrow H^1(O_Y(m))$ is injective for all m . Serre's Vanishing Theorem implies $H^1(O_Y(m)) = 0$ for all m . Q.E.D.

THEOREM 2.3. Let Y be a nonsingular surface in $G \subset \mathbf{P}^5$ of degree $d \leq 8$ not the projection of a surface in $\mathbf{P}^6$, and let (a, b) be its class in the Chow ring $A(G)$ of G . Then one of the following holds:

- (i) $d = 1$, $(a, b) = (1, 0)$, $Y = \mathbf{P}^2$;
- (ii) $d = 2$, $(a, b) = (1, 1)$, $Y = F_0$;
- (iii) $d = 3$, $(a, b) = (2, 1)$, $Y = F_1$,
- (iv) $d = 4$, and either $(a, b) = (2, 2)$ and $Y = F_0, F_2$, or the del Pezzo S_4 , or else $(a, b) = (1, 3)$ and $Y =$ the Veronese surface;
- (v) $d = 5$, $(a, b) = (2, 3)$, and either $Y = F_e$ with 3 points blown up, or $Y = F_e$ with 7 points blown up;
- (vi) $d = 6$, $(a, b) = (3, 3)$ and either $Y = F_e$ with 6 points blown up, or $Y = G \cap \mathbf{P}^4 \cap S_3$ and is a K3 surface;
- (vii) $d = 7$, $Y = F_e$ with 8 points blown up, or Y is the cubic surface with 5 points blown up, or $K^2 = -12 + 6p_a$;
- (viii) $d = 8$, $(a, b) = (4, 4)$, and either Y is the complete intersection of three quadrics, $Y = F_0$ with 10 points blown up, or $Y = G \cap \mathbf{P}^4 \cap S_4$ is a surface of general type.

PROOF. (i) and (ii) are obvious.

(iii) If $d = 3$, then $Y \subset \mathbf{P}^4$, $g(C) = 0$, $HK = -5$, and by Proposition 1.4, $p_a = 0$. By (2), $K^2 = 8$ and Y is geometrically ruled. From (4) we get $e = 1$, $m = 1$, $n = 2$.

(iv) If $d = 4$ and $Y \not\subset \mathbf{P}^4$, then by Lemma 2.1, $g(C) = 0$, $HK = -6$, and $p_a = 0$. If $(a, b) = (1, 3)$, then by (2), $K^2 = 9$. If $(a, b) = (2, 2)$, then $K^2 = 8$. If $Y \subset \mathbf{P}^4$, then by Proposition 1.2(ii), Y is a complete intersection of two quadrics in $\mathbf{P}^4$, hence a del Pezzo S_4 , with $(a, b) = (2, 2)$.

(v) If $d = 5$ and $Y \not\subset \mathbf{P}^4$, then $g(C) = 1$ and $KH = -5$. By Proposition 1.4, Y is ruled and by Lemma 2.2, $p_a = 0$. Hence from (2), $K^2 = 7$ or 5. From Proposition 1.3, $m \leq 3$. Hence by (4), $X^2 = 5 + r = 8$, $K^2 = 5$, and $(a, b) = (2, 3)$, $m = 2$, $e = n - 2$.

But if $Y \subset \mathbf{P}^4$, $g(C) = 2$, $KH = -3$, Y is ruled, and hence by Lemma 2.2, $p_a = 0$. From Proposition 1.3 we have $m \leq 3$. If $m = 2$, then $e = n - 3$, and if $m = 3$, then $3e = 2(n - 2)$. In either case, from (2) and (4) we get $r = 7$ and $(a, b) = (2, 3)$.

(vi) If $d = 6$ and $Y \not\subset \mathbf{P}^4$, then $g(C) = 2$ and Y is ruled, hence from Lemma 2.2, $p_a = 0$. By Proposition 1.3, $m \leq 3$. If $m = 2$, then $e = n - 3$. If $m = 3$, then $3e = 2(n - 2)$. In either case from (2) and (4) we get $r = 6$.

But if $Y \subset \mathbf{P}^4$, by Proposition 1.2(ii), Y is a complete intersection, $\omega_Y = \mathcal{O}_Y$, Y is a $K3$ surface, and $(a, b) = (3, 3)$.

(vii) If $d = 7$ and $Y \not\subset \mathbf{P}^4$, then $g(C) = 3$, $KH = -3$, and Y is a ruled surface with $p_a = 0$ by Lemma 2.2. Hence by (2), $K^2 = 6$ or 2 or 0 . From Proposition 1.3, $m \leq 4$. From (4) one can easily show that $m \neq 2$. If $m = 3$, then $X^2 = 15$ and $K^2 = 0$ (and $3e = 2n - 5$). If $m = 4$, let $r = s + t$, where s is the number of points in X with $r_p = 1$, and t is the number of points in X with $r_p = 2$. Then $X^2 = 7 + s + 4t = 4(2n - 4e)$, and $g(X) = 3 + t$. Hence from (4) we get $27 = 3s + 4t$, and $s = 5$, $t = 3$. Hence $2e = n - 3$, $K^2 = 0$, and $(a, b) = (3, 4)$.

But if $Y \subset \mathbf{P}^4$, then $g(C) = 6$ or 4 . If $g(C) = 4$, then by [10 and 3, Theorem 3.1], $p_a = 0$, hence $K^2 = -2$, and Y is the cubic surface with 5 points blown up. If $g(C) = 6$, then $K^2 = -12 + 6p_a$.

(viii) If $d = 8$ and $Y \not\subset \mathbf{P}^4$, then $g(C) \geq 4$. If $g(C) = 4$, then by [3, Theorem 3.1], $p_a = 0$ and $K^2 = -2$, hence $(a, b) = (4, 4)$ and Y is the blowing up of a nonsingular quadric surface at 10 points. If $g(C) \geq 5$, then the very ample divisor H on C is special. Moreover $\dim |H| = \deg(H)/2$. Hence by [5, p. 343], $H = K_C$, the canonical divisor on C , C is not hyperelliptic, and C is a complete intersection. Therefore Y is a complete intersection of three quadrics in $\mathbf{P}^5$, $\omega_Y = \mathcal{O}_Y$, and by (2), $(a, b) = (4, 4)$.

But if $Y \subset \mathbf{P}^4$ then by Proposition 1.2, Y is a complete intersection $Y = G \cap \mathbf{P}^4 \cap S_4$, $K = H$, $p_a = 5$, Y is a surface of general type, and $(a, b) = (4, 4)$. Q.E.D.

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