THE GAUSS MAP AND NONHOLOMORPHIC HARMONIC MAPS

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ABSTRACT. J. Eells and J. C. Wood [2] gave the fundamental method of construction of certain harmonic maps from Riemann surfaces to complex projective spaces. In this paper, we will construct nonholomorphic harmonic maps of Kaehler manifolds to complex Grassmann manifolds as the Gauss maps for holomorphic immersions of Kaehler manifolds to complex projective spaces.

1. Introduction. Let $\mathbf{P}^N$ be a complex projective N -space. Let $L \rightarrow \mathbf{P}^N$ be the canonical line bundle over $\mathbf{P}^N$. Let $\underline{\mathbf{C}}^{N+1}$ denote the trivial bundle over $\mathbf{P}^N$ with fibre $\mathbf{C}^{N+1}$. Then the holomorphic tangent bundle $T'\mathbf{P}^N$ of $\mathbf{P}^N$ is isomorphic to $L^* \otimes L^\perp$, where L^* is the dual of L and $L^\perp$ is the orthogonal complement of L in $\underline{\mathbf{C}}^{N+1}$. Using this isomorphism, for any point y of $\mathbf{P}^N$, we can regard a linear subspace of $T'_y\mathbf{P}^N$ as a linear subspace of $\mathbf{C}^{N+1}$. Let $f: M \rightarrow \mathbf{P}^N$ be a holomorphic immersion of a Kaehler n -space M to $\mathbf{P}^N$. Since $df(T'_xM)$, $x \in M$, is viewed as a complex n -subspace of $\mathbf{C}^{N+1}$, we can define the Gauss map g of M to a flag manifold $F_{n,n+1} = U(N+1)/U(n) \times U(N-n) \times U(1)$ by $g(x) = (df(T'_xM), \{f(x)\})$, where $\{f(x)\}$ is the complex line determined by $f(x)$. Let π_1 (resp. π_2) be the projection of $F_{n,n+1}$ to $F_n = U(N+1)/U(n) \times U(N-n+1)$ (resp. $F_{n+1} = U(N+1)/U(n+1) \times U(N-n)$). Put $g_i = \pi_i g$ ($i = 1, 2$), these will also be called Gauss maps. Let V be a simply connected complete N -space of constant holomorphic curvature. Nishikawa [6] defined the Gauss map for a holomorphic immersion of a Kaehler manifold into V . In the case $V =$ a complex projective space, the Gauss map of his sense is the same as the map g_2 in our sense. He showed that his Gauss map is antiholomorphic (-holomorphic) [6]. We show that if an immersion is not totally geodesic, the corresponding maps g and g_1 are not $\pm$ holomorphic (Proposition 1). But they are always harmonic (Theorem 5). Hence for every holomorphic immersion $f: M \rightarrow \mathbf{P}^N$ which is not totally geodesic, we can construct harmonic mappings $g: M \rightarrow F_{n,n+1}$ and $g_1: M \rightarrow F_n$ which are not $\pm$ holomorphic. Let M be a Riemann surface. Then it is easy to show that our map $\bar{g}_1: M \rightarrow F_n$ is the same as Eells-Wood's map ϕ produced in [2, Proposition 3.10 (with $r = 1$)]. Recently S. Erdem and J. C. Wood [3] constructed all isotropic harmonic maps from a Riemann surface into a complex Grassmann manifold.

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2. Kaehler immersions. It will be agreed that our indices have the following ranges throughout this paper; $0 \leq A, B, C \leq N$, $1 \leq a, b, c \leq N$, $1 \leq i, j, k \leq n$, $n+1 \leq r, s, t \leq N$, $0 \leq i^*, j^*, k^* \leq n$, $n+1 \leq r^*, s^*, t^* \leq N$ or $r^*, s^*, t^* = 0$.

Let $F\mathbf{P}^N$ be the bundle of unitary frames of $\mathbf{P}^N$. Let $F^*\mathbf{P}^N$ be the set of elements $b = (y, e_0, e_1, \dots, e_N)$ such that $e_0 \in C^{N+1}$, $\|e_0\| = 1$, $d\pi(e_0) = 0$ and $(y, e_1, \dots, e_N) \in F\mathbf{P}^N$, where $\pi: C^{N+1} \rightarrow \mathbf{P}^N$ is the natural projection. Then $F^*\mathbf{P}^N$ is a principal bundle over $\mathbf{P}^N$ with group $U(1) \times U(N)$. Since $T'_y\mathbf{P}^N \simeq L_y^* \times L_y^\perp$, each e_a is regarded as a linear map $e_a: L_y \rightarrow L_y^\perp$. Moreover, by identifying e_a with $e_a(e_0)$, we view $F^*\mathbf{P}^N$ as a subspace of $\mathbf{P}^N \times U(N+1)$.

Let ϕ_B^A be the Maurer-Cartan forms of the unitary group $U(N+1)$. Then $\phi_B^A + \bar{\phi}_A^B = 0$ and $d\phi_B^A = \Sigma \phi_B^C \wedge \phi_C^A$. Let $\rho: \mathbf{P}^N \times U(N+1) \rightarrow U(N+1)$ be the natural projection. Denote by θ^a (resp. θ_b^a) the restriction of $\rho^*\phi_0^a$ (resp. $\rho^*\phi_b^a - \delta_b^a \rho^*\phi_0^0$) to $F^*\mathbf{P}^N$. Then the Kaehler metric on $\mathbf{P}^N$ is given locally by $d\sigma^2 = 2\Sigma \theta^a \bar{\theta}^a$ and the structure equations in $\mathbf{P}^N$ are written as (cf. [6])

$$(1) \quad \begin{cases} d\theta^a = \Sigma \theta^b \wedge \theta_b^a, \\ d\theta_b^a = \Sigma \theta_b^c \wedge \theta_c^a + \theta^a \wedge \bar{\theta}^b + \delta_b^a \Sigma \theta^c \wedge \bar{\theta}^c. \end{cases}$$

Let M be a Kaehler n -manifold isometrically immersed into $\mathbf{P}^N$ by a holomorphic mapping $f: M \rightarrow \mathbf{P}^N$. Let FM be the bundle of unitary frames on M . Let F^*M denote the set of elements $b = (x, e_0, \dots, e_N)$ such that $(x, e_1, \dots, e_n) \in FM$ and $(f(x), e_0, df(e_1), \dots, df(e_n), e_{n+1}, \dots, e_N) \in F^*\mathbf{P}^N$. Then F^*M is a principal bundle over M with group $U(n) \times U(N-n) \times U(1)$. The natural immersion $\tilde{f}: F^*M \rightarrow F^*\mathbf{P}^N$ is defined by $\tilde{f}(b) = (f(x), e_0, df(e_1), \dots, df(e_n), e_{n+1}, \dots, e_N) \in F^*(\mathbf{P}^N)$. On putting $\omega^a = \tilde{f}^*\theta^a$ and $\theta_b^a = \tilde{f}^*\theta_b^a$, we have $\omega^r = 0$ and we can put $\omega_k^r = \Sigma A_{kj}^r \omega^k$, where A_{kj}^r are regarded as the components of the second fundamental form of $f: M \rightarrow \mathbf{P}^N$. The Kaehler metric on M is given locally by $ds^2 = 2\Sigma \theta^i \bar{\theta}^i$. From (1), we obtain the following structure equations in M

$$d\omega^i = \Sigma \omega^k \wedge \omega_k^i, \quad d\omega_j^i = \Sigma \omega_j^k \wedge \omega_k^i + \Sigma (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl} - \Sigma A_{jk}^r \bar{A}_{il}^r) \omega^k \wedge \bar{\omega}^l.$$

Let $-2i\Sigma R_{i\bar{k}} \omega^i \wedge \bar{\omega}^k$ be the Ricci form of M (in this paper, $\bar{i}, \bar{j}, \bar{k}$ run from $\bar{1}$ to $\bar{n}$). Then it holds

$$(2) \quad \Sigma A_{ij}^r \bar{A}_{ik}^r = (n+1)\delta_{jk} - R_{jk}.$$

3. The Gauss map. Let $\tilde{g}$ be the composition of the natural immersion

$$\tilde{f}: F^*M \rightarrow F^*\mathbf{P}^N \subset \mathbf{P}^N \times U(N+1)$$

and the natural projection

$$\rho: \mathbf{P}^N \times U(N+1) \rightarrow U(N+1).$$

Then $\tilde{g}$ is viewed as a bundle map

$$\begin{array}{ccc} F^*M & \xrightarrow{\tilde{g}} & U(N+1) \\ \downarrow & & \downarrow \\ M & \rightarrow & F_{n,n+1} \end{array}$$

The corresponding base map g is just the Gauss map given in §1. Let g_1 and g_2 also be as in §1.

EXAMPLES. (1) Let $f: \mathbf{P}^N \rightarrow \mathbf{P}^N$ be an imbedding of $\mathbf{P}^n$ as a projective subspace of $\mathbf{P}^N$. Let f' be the corresponding map of the set of linear subspaces of $\mathbf{C}^{n+1}$ to the set of linear subspaces of $\mathbf{C}^{N+1}$. Then g_2 is a constant map $g_2(x) = f'(\mathbf{C}^{n+1}) \in F_{n+1}$, $x \in \mathbf{P}^n$. For $x \in \mathbf{P}^n$, let $x^\perp$ be the polar space of x . Then the map f' defines the canonical imbedding f'' of a Grassmann manifold $G(n, 1)$ to $F_n = G(n, N - n + 1)$. Then $g_1(x) = f''(x) \in F_n$, $x \in \mathbf{P}^n$.

(2) Let $f: Q_{N-1} \rightarrow \mathbf{P}^N$ be the canonical imbedding of the hyperquadric in $\mathbf{P}^N$. Let $\bar{x}$ be the complex conjugate line of a complex line $x \in \mathbf{P}^N$. Then $g_1(x) = (x \cup \bar{x})^\perp \in F_{N-1}$ and $g_2(x) = \bar{x}^\perp \in F_n$.

We introduce a quadratic differential form $d\Sigma^2 = 2\Sigma\phi_0^i\bar{\phi}_0^i + 2\Sigma\phi_r^i\bar{\phi}_r^i$ and a complex structure J on the flag manifold $F_{n,n+1}$ by $J\phi_0^i = i\phi_0^j$, $J\phi_r^i = i\phi_r^j$. Then $(d\Sigma^2, J)$ gives a Hermitian structure of $F_{n,n+1}$. From the definition of the Gauss map g , we can see

$$(3) \quad g^*(\phi_0^i) = \omega^i, \quad g^*(\phi_r^0) = 0, \quad g^*(\phi_r^i) = -\Sigma\bar{A}_{ij}^r\bar{\omega}^j.$$

A Kaehler metric $d\Sigma_1^2$ (resp. $d\Sigma_2^2$) on F_n (resp. F_{n+1}) is given by $d\Sigma_1^2 = 2\Sigma\phi_r^i\bar{\phi}_r^i$ (resp. $d\Sigma_2^2 = 2\Sigma\phi_r^i\bar{\phi}_r^{i*}$) and a complex structure J_1 (resp. J_2) on F_n (resp. F_{n+1}) is given by $J_1\phi_r^i = i\phi_r^j$ (resp. $J_2\phi_r^i = i\phi_r^{j*}$). We have also

$$(4) \quad g_1^*(\phi_0^i) = \omega^i, \quad g_1^*(\phi_r^i) = -\Sigma\bar{A}_{ij}^r\bar{\omega}^j,$$

$$(5) \quad g_2^*(\phi_r^0) = 0, \quad g_2^*(\phi_r^i) = -\Sigma\bar{A}_{ij}^r\bar{\omega}^j.$$

Using (2), we can show

$$(6) \quad g^*(d\Sigma^2) = g_1^*(d\Sigma_1^2) = ds^2 + \Sigma A_{ij}^r \bar{A}_{ik}^r \omega^j \bar{\omega}^k = (n+2)ds^2 - \Sigma R_{jk} \bar{\omega}^k \bar{\omega}^j,$$

$$(7) \quad g_2^*(d\Sigma_2^2) = \Sigma A_{ij}^r \bar{A}_{ik}^r \omega^j \bar{\omega}^k = (n+1)ds^2 - \Sigma R_{jk} \bar{\omega}^k \bar{\omega}^j.$$

Nishikawa [6] proved that g_2 is antiholomorphic and that g_2 is a constant map iff the immersion f is totally geodesic. From (6) it follows

PROPOSITION 1. *The immersion f is totally geodesic iff g is isometric and holomorphic, and this holds iff g_1 is isometric and holomorphic.*

As the complex version of some Obata's results in [5], we get

PROPOSITION 2. *Let $f: M \rightarrow \mathbf{P}^N$ be a holomorphic and isometric immersion. The following are equivalent: (1) M Einstein; (2) g a homothety or constant map; (3) g_1 a homothety or constant map; (4) g_2 a homothety or constant map.*

4. The harmonic Gauss map. Put

$$\phi_{js}^{i*} r_s^* = \phi_{js}^{i*} \delta_s^{r*} - \delta_{js}^{i*} \phi_r^{s*}.$$

Then the structure equations in the Kaehler manifold F_n (resp. F_{n+1}) are given by

$$d\phi_r^i = -\Sigma\phi_{js}^{i*} \wedge \phi_s^j \quad (\text{resp. } d\phi_r^{i*} = -\Sigma\phi_{js}^{i*} \wedge \phi_s^{j*}).$$

On the other hand, the structure equations in $F_{n,n+1}$ are given by

$$\begin{aligned} d\phi_0^i &= -\Sigma\phi_{jr}^{i0} \wedge \phi_r^{j*}, \\ d\phi_r^0 &= -\Sigma\phi_{jr}^{0r} \wedge \phi_r^{j*}, \\ d\phi_r^j &= -\Sigma\phi_{js}^{ir} \wedge \phi_s^{j*} - \Sigma\phi_{0s}^{ir} \wedge \phi_s^0 + \phi_0^i \wedge \phi_r^0. \end{aligned}$$

Hence the Hermitian structure $(d\Sigma^2, J)$ is not Kaehlerian. Let $B_j^{i0}, B_j^{i0}, B_j^{0r}, B_j^{0r}, B_j^{ir}, B_j^{ir}$ be the coefficients of the Jacobian matrix $J(g)$, that is, $g^*(\phi_0^i) = \Sigma B_j^{i0}\omega^j + \Sigma B_j^{i0}\bar{\omega}^j$ and so on. Then from (3), it follows that $B_j^{i0} = \delta_j^i, B_j^{ir} = -\bar{A}_{ij}^r$ and any other components vanish. We will calculate the covariant derivative of the Jacobian $J(g)$. We can put [1]

$$\begin{aligned} \Sigma B_{jk}^{i0}\omega^k + \Sigma B_{jk}^{i0}\bar{\omega}^k &= dB_j^{i0} - \Sigma B_k^{i0}\omega_j^k + \Sigma B_j^{k0}g^*(\phi_{k0}^i) \\ &\quad + \Sigma B_j^{0r}g^*(\phi_{0r}^i) + \Sigma B_j^{kr}g^*(\phi_{kr}^i) = 0. \end{aligned}$$

Hence $B_{jk}^{i0} = B_{jk}^{i0} = 0$. Similarly we can show that all $B_{jk}^{i0}, B_{jk}^{i0}, B_{jk}^{0r}, B_{jk}^{0r}, B_{jk}^{ir}, B_{jk}^{ir}$ vanish. By exterior differentiation of the last equation in (3), we get

$$(\Sigma B_{jk}^{ir}\omega^k + \Sigma B_{jk}^{ir}\bar{\omega}^k) \wedge \bar{\omega}^j = 0.$$

Thus we obtain

LEMMA 3. *The only nonvanishing components of the covariant derivative of the components of the Jacobian matrix $J(g)$ are B_{jk}^{ir} .*

Similarly we put $g^*(\phi_r^{i*}) = \Sigma C_j^{ir}\omega^j + \Sigma C_j^{ir}\bar{\omega}^j$. Then $C_j^{i0} = \delta_j^i, C_j^{ir} = 0, C_j^{i0} = 0, C_j^{ir} = -\bar{A}_{ij}^r$, and we can prove

LEMMA 4. *The only nonvanishing components of the covariant derivative of the components of the Jacobian matrix $J(g_1)$ are C_{jk}^{ir} .*

From Lemma 3, it follows that $\Sigma B_{jj}^{i0} = 0, \Sigma B_{jj}^{0r} = 0$ and $\Sigma B_{jj}^{ir} = 0$. Hence g is harmonic (see [1]). Similarly g_1 is harmonic. Thus we get

THEOREM 5. *Let $f: M \rightarrow \mathbf{P}^N$ be a holomorphic and isometric immersion of a Kaehler n -manifold M . Then the corresponding Gauss maps $g: M \rightarrow F_{n,n+1}$ and $g_1: M \rightarrow F_n$ are harmonic.*

Let $\pi_1: F_{n,n+1} \rightarrow F_n$ be the natural projection. Then we have $\pi_1^*(\phi_r^{i*}) = \phi_r^{i*}$. Combining this with (3), we get (the definition of π_1 -horizontal maps is found in [2, 4]),

PROPOSITION 6. *The Gauss map g is π_1 -horizontal.*

Let M be a compact complex manifold and $L \rightarrow M$ be a holomorphic line bundle. Every linear system of divisors for $L \rightarrow M$ defines a holomorphic mapping of M to $\mathbf{P}^N$. In particular, the Kodaira imbedding theorem asserts that if $L \rightarrow M$ is a positive line bundle, there is an imbedding of M to $\mathbf{P}^N$. Hence using Theorem 5, we can construct a lot of nonholomorphic harmonic maps of M to complex Grassmann manifolds.

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