ON THE DERIVATION OF x^n IN A RING

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ABSTRACT. The main result of this paper gives conditions which ensure that an additive map of a ring R into itself which satisfies a generalisation of the rule $dx^n/dx = nx^{n-1}$ is a derivation on R .

It is well known that the product rule for derivatives can be proved using only the additivity of the differentiation process, the Chain Rule, and the formula for the derivative of the map $x \rightarrow x^2$ (see Lemma 1). In this note, we show how this generalises to a characterisation of derivations on certain rings.

Let R be a ring, and recall that a *derivation* on R is an additive mapping $D: R \rightarrow R$ such that

$$D(xy) = xD(y) + D(x)y \quad (x, y \in R).$$

It is easy to show that for each integer $n \geq 2$, such a map has the n th power property:

$$D(x^n) = \sum_{j=1}^n x^{n-j}D(x)x^{j-1} \quad (x \in R).$$

(We adopt the convention that $x^0y = y = yx^0$ for all x, y in R .) Conversely, if $D: R \rightarrow R$ is an additive map with the n th power property for some $n \geq 2$, under what conditions is D a derivation on R ?

One almost trivial case when D is a derivation is dealt with in our first result.

LEMMA 1. *If R is a commutative ring with no 2-torsion, and the additive map $D: R \rightarrow R$ has the second power property, then D is a derivation on R .*

PROOF. For all x, y in R ,

$$\begin{aligned} 2(x+y)(D(x) + D(y)) &= D((x+y)^2) = D(x^2) + 2D(xy) + D(y^2) \\ &= 2xD(x) + 2D(xy) + 2yD(y). \end{aligned}$$

It readily follows that $D(xy) = xD(y) + D(x)y$. $\square$

A more general answer to the question above was given by Herstein [1]. For convenience, we state as much of Herstein's result as we shall need.

THEOREM 1. *Let R be a prime ring that is not of characteristic 2, and let $D: R \rightarrow R$ be an additive map with the 2nd power property. Then D is a derivation on R .*

For our main result we need two more lemmas.

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LEMMA 2. Let R be a ring, D an additive map of R into itself, and y an element of R such that both y and $D(y)$ are in the center of R . For each x in R and each positive integer k , define

$$f_k(x) \equiv \sum_{j=1}^k x^{k-j} D(x) x^{j-1}.$$

Then

$$(1) \quad f_n(x+y) = \sum_{j=1}^n \binom{n}{j} y^{n-j} f_j(x) + n(x+y)^{n-1} D(y) \quad (n \geq 1).$$

PROOF. We proceed by induction on n . In the case $n = 1$, equation (1) is just a particular case of the additivity of D . Now suppose that we have proved (1) for $n = k$. Then

$$\begin{aligned} f_{k+1}(x+y) &= (x+y)f_k(x+y) + D(x+y)(x+y)^k \\ &= (x+y) \left[\sum_{j=1}^k \binom{k}{j} y^{k-j} f_j(x) + k(x+y)^{k-1} D(y) \right] \\ &\quad + D(x)(x+y)^k + D(y)(x+y)^k \\ &= \sum_{j=1}^k \binom{k}{j} x f_j(x) y^{k-j} + \sum_{j=1}^k \binom{k}{j} y^{k+1-j} f_j(x) \\ &\quad + k(x+y)^k D(y) + D(x)(x+y)^k + (x+y)^k D(y) \\ &= \sum_{j=1}^k \binom{k}{j} [f_{j+1}(x) - D(x)x^j] y^{k-j} + \sum_{j=1}^k \binom{k}{j} y^{k+1-j} f_j(x) \\ &\quad + (k+1)(x+y)^k D(y) + D(x) \sum_{j=0}^k \binom{k}{j} x^j y^{k-j} \\ &= f_{k+1}(x) + \sum_{j=2}^k \left[\binom{k}{j-1} + \binom{k}{j} \right] y^{k+1-j} f_j(x) + k y^k f_1(x) \\ &\quad + (k+1)(x+y)^k D(y) + D(x) y^k \\ &= f_{k+1}(x) + \sum_{j=2}^k \binom{k+1}{j} y^{k+1-j} f_j(x) \\ &\quad + (k+1) y^k f_1(x) + (k+1)(x+y)^k D(y) \\ &= \sum_{j=1}^{k+1} \binom{k+1}{j} y^{k+1-j} f_j(x) + (k+1)(x+y)^k D(y). \quad \square \end{aligned}$$

In the proofs below, for a fixed integer $n \geq 2$ we need the observation that the $(n-1)$ -by- $(n-1)$ matrix A with (k, j) th entry $k^j \binom{n}{j}$ has determinant

$$\delta \equiv \det A = \left(\prod_{k=1}^{n-1} \binom{n}{k} \right) (n-1)! \prod_{1 \leq i < j \leq n-1} (i-j).$$

LEMMA 3. Let R be a ring, $n \geq 2$ an integer, $D: R \rightarrow R$ an additive map with the n th power property, and y an element of R such that both y and $D(y)$ are central. Suppose that for each x in R , $\delta x = 0$ entails $x = 0$. Then

$$(2) \quad D(x^k y^{n-k}) = y^{n-k} \sum_{j=1}^k x^{k-j} D(x) x^{j-1} + (n-k) x^k y^{n-k-1} D(y)$$

whenever $x \in R$ and $k = 1, 2, \dots, n-1$.

PROOF. For each x in R and each positive integer k , define

$$f_k(x) \equiv \sum_{j=1}^k x^{k-j} D(x) x^{j-1}.$$

By the n th power property and Lemma 2,

$$D((x+y)^n) = f_n(x+y) = f_n(x) + \sum_{j=1}^{n-1} \binom{n}{j} y^{n-j} f_j(x) + n(x+y)^{n-1} D(y).$$

On the other hand,

$$\begin{aligned} D((x+y)^n) &= D\left(\sum_{j=0}^n \binom{n}{j} y^{n-j} x^j\right) \\ &= D(x^n) + \sum_{j=1}^{n-1} \binom{n}{j} D(y^{n-j} x^j) + D(y^n). \end{aligned}$$

Equating these expressions for $D((x+y)^n)$, we obtain

$$(3) \quad \sum_{j=1}^{n-1} \binom{n}{j} [D(y^{n-j} x^j) - y^{n-j} f_j(x)] + D(y^n) - n(x+y)^{n-1} D(y) = 0.$$

Since y commutes with $D(y)$, we have $D(y^n) = ny^{n-1}D(y)$, and therefore,

$$\begin{aligned} n(x+y)^{n-1} D(y) - D(y^n) &= n \sum_{j=1}^{n-1} \binom{n-1}{j} x^j y^{n-1-j} D(y) \\ &= \sum_{j=1}^{n-1} (n-j) \binom{n}{j} x^j y^{n-1-j} D(y). \end{aligned}$$

From this and (3) we have

$$\sum_{j=1}^{n-1} \binom{n}{j} [D(x^j y^{n-j}) - y^{n-j} f_j(x) - (n-j) x^j y^{n-1-j} D(y)] = 0.$$

Replacing x by kx , we now obtain

$$\sum_{j=1}^{n-1} k^j \binom{n}{j} [D(x^j y^{n-j}) - y^{n-j} f_j(x) - (n-j) x^j y^{n-1-j} D(y)] = 0$$

$$(k = 1, 2, \dots, n-1).$$

Since the matrix of these equations has (k, j) th entry $k^j \binom{k}{j}$, for $k = 1, 2, \dots, n-1$, we have

$$\delta \left[D(x^j y^{n-j}) - y^{n-j} f_j(x) - (n-j)x^j y^{n-1-j} D(y) \right] = 0$$

from which the desired conclusion follows. $\square$

THEOREM 2. *Let R be a ring, $n \geq 2$ an integer, and $D: R \rightarrow R$ an additive mapping with the n th power property. Then D is a derivation under each of the following circumstances:*

- (i) R is commutative with an identity e , and for each x in R , $\delta x = 0$ entails $x = 0$,
- (ii) R is a prime ring with an identity e and with characteristic c , where $c = 0$ or $c > n$,
- (iii) R is a commutative integral domain with characteristic c , where $c = 0$ or $c > n$.

PROOF. First note that in all these circumstances we have $x = 0$ whenever $x \in R$ and $\delta x = 0$. In cases (ii) and (iii), this is clear if $c = 0$; while if c is prime, it follows from the fact that as $c > n$, c does not divide δ .

Consider cases (i) and (ii). Since $D(e) = D(e^n) = ne^{n-1}D(e) = nD(e)$, we have $(n-1)D(e) = 0$, and therefore $\delta D(e) = 0$. Hence $D(e) = 0$. If we now apply Lemma 3, taking $y = e$ and $k = 2$ in (2), we see that D satisfies the 2nd power property. It follows from Lemma 1 in case (i), and from Theorem 1 in case (ii), that D is a derivation.

Now consider case (iii), and apply Lemma 3. Taking $k = n-1$ in (2), we see that for arbitrary x and y in R ,

$$\begin{aligned} (4) \quad D(x^{n-1}y) &= y \sum_{j=1}^{n-1} x^{n-1-j} D(x) x^{j-1} + x^{n-1} D(y) \\ &= (n-1)x^{n-2}yD(x) + x^{n-1}D(y). \end{aligned}$$

With $y = x^{n+1}$, we obtain

$$\begin{aligned} nx^{2n-2}D(x^2) &= D((x^2)^n) = D(x^{n-1}x^{n+1}) \\ &= (n-1)x^{2n-1}D(x) + x^{n-1}D(x^{n+1}). \end{aligned}$$

Also, putting $y = x^2$ in (4) gives

$$D(x^{n+1}) = (n-1)x^n D(x) + x^{n-1}D(x^2).$$

Hence

$$nx^{2n-2}D(x^2) = (n-1)x^{2n-1}D(x) + x^{n-1}[(n-1)x^n D(x) + x^{n-1}D(x^2)]$$

and so

$$(n-1)x^{2n-2}[D(x^2) - 2xD(x)] = 0.$$

Since R has no zero divisors, it follows that $D(x^2) = 2xD(x)$. (Note that $D(0) = 0$, as D is additive.) As x is arbitrary, D has the 2nd power property, and is therefore a derivation, by Lemma 1. $\square$

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REFERENCES

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