PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 1, January 1984

CO-WELL-POWERED REFLECTIVE SUBCATEGORIES

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ABSTRACT. A full isomorphism-closed subcategory $\mathcal{A}$ of a complete well-powered and co-well-powered category $\mathcal{C}$ is both co-well-powered (in its own right) and reflective in $\mathcal{C}$ if and only if

(a) $\mathcal{A}$ is closed in $\mathcal{C}$ under the formation of (U-small-indexed) limits, and

(b) the epi-reflective hull $\mathcal{B}$ of $\mathcal{A}$ in $\mathcal{C}$ is co-well-powered.

A full isomorphism-closed subcategory $\mathcal{Y}$ of a well-powered and co-well-powered complete category $\mathcal{X}$ is epi-reflective in $\mathcal{X}$ (i.e. full, isomorphism-closed and reflective with epimorphic reflection morphisms) if and only if $\mathcal{Y}$ is stable in $\mathcal{X}$ under the formation of products (indexed over U-small sets) and under extremal subobjects—where U denotes a fixed universe (cf. [**2**, p. 87; **6**, p. 1276; **7**, p. 356]). Indeed, a slightly weaker requirement suffices to ensure the epi-reflectiveness of $\mathcal{Y}$ in $\mathcal{X}$: $\mathcal{Y}$ is closed under (U-small-indexed) products and *strongly closed* under difference kernels, i.e. whenever

$$A \underset{f}{\rightarrow} B \overset{g}{\underset{h}{\rightrightarrows}} C$$

is a difference kernel (or equalizer) in $\mathcal{X}$ with $B \in \mathrm{Ob}\,\mathcal{Y}$, then $A \in \mathrm{Ob}\,\mathcal{Y}$ [**3**, 10.2.1].

The smallest epi-reflective subcategory $\mathcal{D}$ of a well-powered and co-well-powered complete category $\mathcal{C}$ containing a given subcategory $\mathcal{X}$ of $\mathcal{C}$, the "*epi-reflective hull*" $\mathcal{D}$ of $\mathcal{X}$ in $\mathcal{C}$, consists of all $\mathcal{C}$-objects which are (domains of) extremal subobjects of products (over a U-small index set) of members of Ob $\mathcal{X}$. Every full isomorphism-closed reflective subcategory $\mathcal{A}$ of a well-powered and co-well-powered complete category $\mathcal{C}$ is both mono-reflective and (consequently) epi-reflective in the epi-reflective hull $\mathcal{B}$ of $\mathcal{A}$ in $\mathcal{C}$ [**1**]. Sharpening results in [**1** and **7**], it is observed in [**4**] that *a full subcategory $\mathcal{A}$ of a complete, well-powered and co-well-powered category $\mathcal{C}$ is reflective in $\mathcal{C}$ if* (i) *and* (ii) *are satisfied*:

(i) *$\mathcal{A}$ is stable in $\mathcal{C}$ under the formation of* (U*-small-indexed*) *limits*;

(ii) *the epi-reflective hull $\mathcal{B}$ of $\mathcal{A}$ in $\mathcal{C}$ is co-well-powered.*

(Indeed, a difference kernel

$$X \overset{u}{\rightarrow} Y \overset{v}{\underset{w}{\rightrightarrows}} Z$$

in $\mathcal{B}$ with $Y \in \mathrm{Ob}\,\mathcal{A}$ yields a difference kernel

$$X \overset{u}{\rightarrow} Y \overset{mv}{\underset{mv}{\rightrightarrows}} A$$

for some extremal monomorphism $m\colon Z \rightarrow A$ in $\mathcal{C}$ with $A \in \mathrm{Ob}\,\mathcal{A}$; hence $X \in \mathrm{Ob}\,\mathcal{A}$ by hypothesis. Consequently, $\mathcal{A}$ is epi-reflective in $\mathcal{B}$.)

While (i) is clearly necessary for reflectiveness of $\mathcal{A}$ in $\mathcal{C}$, condition (ii) is not. Here we wish to add the following observations.

Received by the editors December 13, 1982.

1980 *Mathematics Subject Classification*. Primary 18A40.

Key words and phrases. Reflective subcategory, co-well-powered category, epi-reflective hull.

0002-9939/84 $1.00 + $.25 per page

1. *A full subcategory $\mathcal{A}$ of a complete, well-powered and co-well-powered category $\mathcal{C}$ is co-well-powered if the epi-reflective hull $\mathcal{B}$ of $\mathcal{A}$ in $\mathcal{C}$ is also.*

Indeed, the inclusion $\mathcal{A} \to \mathcal{B}$ preserves epimorphisms [**5**, 2.2].

2. *A full isomorphism-closed subcateogry $\mathcal{A}$ of a complete well-powered and co-well-powered category $\mathcal{C}$ is both co-well-powered (in its own right) and reflective in $\mathcal{C}$ if and only if*

(a) *$\mathcal{A}$ is closed in $\mathcal{C}$ under the formation of (U-small-indexed) limits, and*

(b) *the epi-reflective hull $\mathcal{B}$ of $\mathcal{A}$ in $\mathcal{C}$ is co-well-powered.*

PROOF. It remains to establish the necessity of (b). For $X \in \operatorname{Ob} \mathcal{C}$, let $r_X : X \to R(X)$ denote the $\mathcal{A}$-reflection morphism of X. Now let $B \in \operatorname{Ob} \mathcal{B}$. For every $\mathcal{B}$-epimorphism $e \colon B \to Y$ we obtain a commutative square:

$$\begin{array}{ccc} R(B) & \overset{R(e)}{\dashrightarrow} & R(Y) \\ {\scriptstyle r_B}\uparrow & & \uparrow{\scriptstyle r_Y} \\ B & \underset{e}{\dashrightarrow} & Y \end{array}$$

Since r_Y is a $\mathcal{B}$-epimorphism, so is $r_Y e = R(e) r_B$; hence so is $R(e)$. Consequently, $R(e)$ is an epimorphism in $\mathcal{A}$. There are suitable isomorphisms j_e in $\mathcal{A}$ with domain $R(Y)$ so that we obtain a mapping $\varphi \colon e \mapsto j_e R(e)$ from a representative system of $\mathcal{B}$-epimorphisms with domain B into a representative system of $\mathcal{A}$-epimorphisms with domain $R(B)$. The latter set is U-small by hypothesis. Since r_Y is an (extremal) monomorphism and since $\mathcal{C}$ is well-powered, the fibers of this mapping φ (i.e. the inverse images of single elements) are U-small. As a consequence, the domain of φ is also U-small, i.e. $\mathcal{B}$ is co-well-powered.

Necessary and sufficient conditions for a subcategory to be co-well-powered reflective are also given in [**1**, Theorem 3].

NOTE ADDED IN PROOF. A careful examination of the proofs of the results leading to the theorem obtained above shows that the latter can be extended to a complete category $\mathcal{C}$ with a "well-founded" bicategory structure (E, M) [**7**, p. 355] when the epi-reflective hull $\mathcal{B}$ of $\mathcal{A}$ is replaced by the E-reflective hull of $\mathcal{A}$ in $\mathcal{C}$. (This transfers co-well-poweredness from *compact T_2-spaces* to *completely Hausdorff spaces* (with *continuous maps*) as well as to *every* intermediate full subcategory.)

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