TALES ABOUT TAILS¹

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ABSTRACT. Necessary and sufficient conditions are given for a sequence $\{g^{(n)}\}$ to be a sequence of tails of a convergent continued fraction. Some special cases are also studied.

0. Introduction. The problems to be discussed in the present paper all have to do with continued fractions

$$(0.1) \quad K_{n=1}^{\infty} \frac{a_n}{1}, \quad a_n \in \mathbb{C}, a_n \neq 0, n = 1, 2, 3, \dots$$

For any fixed n the extended complex number

$$(0.2) \quad f_n = K_{\nu=1}^n \frac{a_{\nu}}{1}, \quad f_0 = 0,$$

is called the n th approximant of the continued fraction (0.1). In case of convergence the limit

$$(0.3) \quad \lim_{n \rightarrow \infty} f_n = f$$

exists and is called the *value* of the continued fraction (0.1). The word “approximant” is meaningful, and the sequence $\{f_n\}_{n=1}^{\infty}$ of approximants exists regardless of whether or not (0.1) converges.

If, in particular, the continued fraction (0.1), and hence all its tails,

$$(0.4) \quad K_{\nu=n+1}^{\infty} \frac{a_{\nu}}{1},$$

converge (possibly to ∞), then the sequence $\{f^{(n)}\}_{n=0}^{\infty}$ of extended complex numbers, where $f^{(n)}$ is the value of (0.4), $f^{(0)} = f$, is called the sequence of *right tails* of the convergent continued fraction (0.4). Obviously the numbers $f^{(n)}$ satisfy the set of recursion relations

$$(0.5) \quad f^{(n)} = \frac{a_{n+1}}{1 + f^{(n+1)}}, \quad n = 0, 1, 2, \dots,$$

understood such that in the particular case $f^{(n)} = \infty$, we have $f^{(n+1)} = -1$. For a given convergent continued fraction (0.1) (convergence to ∞ being permitted) the

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sequence of right tails is unique. The condition $a_n \neq 0$, $n = 1, 2, 3, \dots$, which is part of the definition, is essential. Any *other* sequence of extended complex numbers satisfying (0.5), i.e. any sequence $\{g^{(n)}\}_{n=0}^\infty$ with $g^{(0)} \neq f^{(0)}$ and such that

$$(0.5') \quad g^{(n)} = \frac{a_{n+1}}{1 + g^{(n+1)}} \quad \text{for all } n \geq 0,$$

is called a sequence of *wrong tails*. Tails of continued fractions (also wrong tails) are of vital importance for modifications of continued fractions. The survey article [13] discusses modifications and their applications, and also contains an extensive bibliography. Newer results are found in [4–7].

In [10, Theorem 2.7] the following question is discussed: Which sequences $\{f_n\}_{n=1}^\infty$ are sequences of approximants for some continued fraction $b_0 + K_{n=1}^\infty a_n/b_n$? The answer is that “almost any sequence is”. The only conditions that have to be satisfied are $f_0 \neq \infty$ and $f_{n-1} \neq f_n$ for all $n \geq 1$. For continued fractions $b_0 + K(a_n/1)$ the conditions are $f_0 \neq \infty$ and $f_{n-1} \neq f_n, f_{n-1} \neq f_{n+1}$ for all $n \geq 1$.

In the present paper we shall raise the corresponding question for tails:

Which sequences $\{g^{(n)}\}_{n=0}^\infty$ are sequences of right tails for some (convergent) continued fraction?

In view of (0.5') we have

$$(0.6) \quad g^{(n)}(1 + g^{(n+1)}) = a_{n+1} \quad \text{for all } n \geq 0.$$

Since all $a_n \neq 0$, we have $(g^{(n)} = 0) \Leftrightarrow (g^{(n+1)} = \infty)$ and $(g^{(n)} = \infty) \Leftrightarrow (g^{(n+1)} = -1)$, but in neither of these cases a_{n+1} is uniquely determined. In the present paper we shall avoid this situation by assuming that $g^{(n)} \neq 0, \infty$ for all $n \geq 0$, and $g^{(n)} \neq -1$ for all $n \geq 1$, in which case the continued fraction is uniquely determined as

$$(0.7) \quad \frac{g^{(0)}(1 + g^{(1)})}{1} + \frac{g^{(1)}(1 + g^{(2)})}{1} + \frac{g^{(2)}(1 + g^{(3)})}{1} + \dots$$

The sequence $\{g^{(n)}\}$ is a sequence of *right tails* for this if and only if (0.7) converges to $g^{(0)}$. This is how the problem will be attacked in the next section.

Why is this problem interesting? The method of acceleration of convergence, as presented in [4 and 5], and further extended in [6], uses auxiliary (convergent) continued fractions, where we need to know the value of all tails. The method thus heavily depends upon the knowledge of such auxiliary continued fractions. If the supply of known continued fractions with known tails increases, then the possibility of using the method also increases. A complete, or even a partial answer to the question raised in the present paper obviously makes it possible to construct continued fractions with known tails. (A different type of investigation of tails is presented in [8].)

1. The main result. It turns out that under the additional conditions mentioned above the question can easily be settled completely.

THEOREM 1. *Let $\{g^{(n)}\}_{n=0}^{\infty}$ be a sequence of complex numbers, all $\neq 0$ and all except possibly $g^{(0)}$ different from -1 . Furthermore, let*

$$(1.1) \quad \kappa_n = -\frac{1 + g^{(n)}}{g^{(n)}} \quad \text{for } n = 1, 2, 3, \dots$$

Then $\{g^{(n)}\}_{n=0}^{\infty}$ is the sequence of right tails for some continued fraction if and only if

$$(1.2) \quad 1 + \kappa_1 + \kappa_1\kappa_2 + \kappa_1\kappa_2\kappa_3 + \dots = \infty.$$

(This theorem is related to [11, Satz 2.45].)

PROOF. From the previous section we know that $\{g^{(n)}\}$ is the sequence of right tails for some continued fraction (which *has to be* (0.7)) if and only if (0.7) converges to $g^{(0)}$. Let in (0.1)

$$(1.3) \quad a_{n+1} = g^{(n)}(1 + g^{(n+1)}), \quad n \geq 0,$$

and let A_n and B_n have the standard meaning, given by

$$(1.4) \quad f_n = \frac{A_n}{B_n}, \quad n \geq 0,$$

and the initial conditions

$$(1.5) \quad A_0 = 0, \quad A_1 = a_1, \quad B_0 = 1, \quad B_1 = 1.$$

From [9, Propositions 1 and 2] we know that

$$(1.6) \quad A_n - B_n g^{(0)} = (-1)^{n-1} \prod_{m=0}^n g^{(m)}$$

and

$$(1.7) \quad B_n = \sum_{m=1}^{n+1} (-1)^{n+1-m} \left(\prod_{\nu=m}^n g^{(\nu)} \right) \left(\prod_{\nu=1}^{m-1} (1 + g^{(\nu)}) \right)$$

(see also [15, p. 47]). If we divide (1.6) by (1.7) and use (1.4) and (1.1) we get

$$(1.8) \quad f_n - g^{(0)} = \frac{-g^{(0)}}{1 + \kappa_1 + \kappa_1\kappa_2 + \dots + \kappa_1\kappa_2 \dots \kappa_n},$$

from which the theorem follows immediately.

REMARK. We may actually squeeze much more information out of (1.8). If we briefly call the series $1 + \kappa_1 + \kappa_1\kappa_2 + \dots$ “the κ series”, the sequence $\{g^{(n)}\}$ merely

“the g -sequence” and let “the continued fraction” mean (0.7), we get some results of the type:

$\nearrow$ Property of g -sequence
 Property of κ -series
 $\searrow$ Property of continued fraction.

We list them in the following way:

κ -series	g -sequence	Continued fraction
converges to ∞	is a sequence of right tails of a continued fraction	converges to $g^{(0)}$
converges to 0	is a sequence of wrong tails of a continued fraction	converges to ∞
converges to $H_0 \neq 0, \infty$	is a sequence of wrong tails of a continued fraction	converges to $g^{(0)}(1 - 1/H_0)$
diverges	is a sequence of wrong tails of a continued fraction	diverges

2. A result on wrong tails. From the remark at the end of the previous section we find that if the κ -series converges to an $H_0 \neq \infty$, then the continued fraction converges (possibly to ∞). Being a convergent continued fraction it has a unique sequence of right tails $f^{(n)}$. If the κ -sequence converges to $H_0 \neq \infty$ we have

$$(2.1) \quad f^{(0)} = g^{(0)}(1 - 1/H_0).$$

Since all $\kappa_n \neq 0$, it follows from the convergence of

$$(2.2') \quad 1 + \kappa_1 + \kappa_1\kappa_2 + \kappa_1\kappa_2\kappa_3 + \dots$$

to an $N_0 \neq \infty$ that all the series

$$(2.2) \quad 1 + \kappa_{n+1} + \kappa_{n+1}\kappa_{n+2} + \kappa_{n+1}\kappa_{n+2}\kappa_{n+3} + \dots$$

converge to numbers $H_n \neq \infty$, and it follows trivially that

$$(2.3) \quad f^{(n)} = g^{(n)}(1 - 1/H_n), \quad n = 0, 1, 2, \dots$$

(Observe that (2.3) also holds if $H_0 = \infty$, in which case all $H_n = \infty$. (2.3) then reduces to $f^{(n)} = g^{(n)}$.)

We thus have a procedure for computing the right tails of a convergent continued fraction (0.1) if we know an arbitrary sequence $\{g^{(n)}\}_{n=0}^{\infty}$ of wrong tails with $g^{(n)} \neq 0, -1, \infty$. Obviously we cannot expect $f^{(n)}$ to be different from 0, -1, ∞ .

On this background the next theorem, giving sufficient conditions for a sequence to be a sequence of *wrong* tails for a convergent continued fraction, is of a certain interest.

THEOREM 2. *Let r be an arbitrary positive number < 1 , and let D_r be the disk*

$$(2.4) \quad D_r = \left\{ w \mid \left| w + \frac{1}{1-r^2} \right| \leq \frac{r}{1-r^2} \right\}.$$

Let $\{g^{(n)}\}_{n=0}^\infty$ be a sequence of complex numbers, all located in D_r . Then $\{g^{(n)}\}_{n=0}^\infty$ is a sequence of wrong tails of a convergent continued fraction.

PROOF. Simple computation shows that

$$g^{(n)} \in D_p \Leftrightarrow |\kappa_n| \leq r.$$

Hence if $g^{(n)} \in D_r$, the series

$$(2.2') \quad 1 + \kappa_1 + \kappa_1 \kappa_2 + \kappa_1 \kappa_2 \kappa_3 + \cdots$$

converges to a number $\leq 1/(1-r) < \infty$, and the statement in the theorem follows from the remarks after Theorem 1.

Theorem 2 is a convergence theorem for continued fractions (0.1), where sufficient conditions for convergence are given in terms of numbers $g^{(n)}$, such that $a_{n+1} = g^{(n)}(1 + g^{(n+1)})$. Related theorems are proved by Scott and Wall, Paydon and Wall, Pringsheim, Perron and Van Vleck (see [15, Theorem 11.1–4]). Also, in these theorems an essential role is played by the sum (2.2') (in a more special case).

3. Examples. In the present section we shall study three special cases in connection with Theorem 1.

(a) *Periodic sequence.* Assume in addition to the conditions of Theorem 1 that the sequence $\{g^{(n)}\}_{n=0}^{\infty}$ is periodic with period k . By studying the partial sums S_{nk+p} , $p = 0, 1, 2, \dots, k-1$ of the κ -series it follows immediately from Theorem 1 that the g -sequence is a sequence of right tails of a convergent continued fraction if and only if

[illegible]

(For $k = 1$ or $k = 2$ the condition to the right is empty.) Before leaving the periodic case we shall look at the transition from wrong tails to right tails in this case. For simplicity we shall restrict ourselves to the case $k = 3$. But even the general case does not offer any problem. We assume, according to the remark after Theorem 1, that the κ -series converges to an $H_0 \neq \infty$. In addition, we shall assume that the continued fraction converges to a finite number, in which case we also know that $H_0 \neq 0$. The κ -series can be rearranged in the following way

$$(1 + \kappa_1 + \kappa_1\kappa_2)(1 + \kappa_1\kappa_2\kappa_3 + (\kappa_1\kappa_2\kappa_3)^2 + \cdots).$$

Since this converges to a finite number $H_0 \neq 0$ we have

$$1 + \kappa_1 + \kappa_1\kappa_2 \neq 0 \quad \text{and} \quad |\kappa_1\kappa_2\kappa_3| < 1,$$

and hence

$$H_0 = \frac{1 + \kappa_1 + \kappa_1\kappa_2}{1 - \kappa_1\kappa_2\kappa_3}.$$

Similarly we get

$$H_1 = \frac{1 + \kappa_2 + \kappa_2\kappa_3}{1 - \kappa_1\kappa_2\kappa_3}, \quad H_2 = \frac{1 + \kappa_3 + \kappa_3\kappa_1}{1 - \kappa_1\kappa_2\kappa_3}.$$

(H_1 or H_2 may possibly be 0.) From (2.3) we find

$$(3.2) \quad \begin{cases} f^{(0)} = g^{(0)} \frac{\kappa_1(1 + \kappa_2 + \kappa_2\kappa_3)}{1 + \kappa_1 + \kappa_1\kappa_2}, \\ f^{(1)} = g^{(1)} \frac{\kappa_2(1 + \kappa_3 + \kappa_3\kappa_1)}{1 + \kappa_2 + \kappa_2\kappa_3}, \\ f^{(2)} = g^{(2)} \frac{\kappa_3(1 + \kappa_1 + \kappa_1\kappa_2)}{1 + \kappa_1 + \kappa_3\kappa_1}. \end{cases}$$

We find similar formulas in the general k -case. For $k = 1$ the formulas reduce to the well-known $f^{(0)} = -(1 + g^{(0)})$, and for $k = 2$ they are $f^{(0)} = -(1 + g^{(1)})$, $f^{(1)} = -(1 + g^{(0)})$ (see [13, p. 64]).

(b) *Positive sequences.* Assume in addition to the conditions on Theorem 1 that all $g^{(n)}$ are positive. Let S_{2m} and S_{2m+1} denote partial sums of the κ -series. Then

$$\begin{aligned} -S_{2m} &= -(1 + \kappa_1 + \kappa_1\kappa_2 + \kappa_1\kappa_2\kappa_3 + \cdots + \kappa_1\kappa_2 \cdots \kappa_{2m-1}) \\ &= \frac{1}{g^{(1)}} + \frac{\kappa_1\kappa_2}{g^{(3)}} + \cdots + \frac{\kappa_1\kappa_2 \cdots \kappa_{2m-2}}{g^{(2m-1)}} > \sum_{k=1}^m \frac{1}{g^{(2k-1)}}. \end{aligned}$$

Similarly, we get $S_{2m+1} > 1 + \sum_{k=1}^m 1/g^{(2k)}$. Hence

THEOREM 3. *If $\{g^{(n)}\}_{n=0}^\infty$ is a sequence of positive numbers such that*

$$(3.3) \quad \sum_{m=1}^\infty \frac{1}{g^{(2m)}} = \infty \quad \text{and} \quad \sum_{m=1}^\infty \frac{1}{g^{(2m+1)}} = \infty,$$

Then $\{g^{(n)}\}_{n=0}^\infty$ is the sequence of right tails for the continued fraction (0.7).

COROLLARY 4. *If $\{g^{(n)}\}_{n=0}^\infty$ is a monotone sequence of positive numbers such that*

$$(3.4) \quad \sum_{n=1}^\infty \frac{1}{g^{(n)}} = \infty,$$

then $\{g^{(n)}\}_{n=0}^\infty$ is the sequence of right tails for the continued fraction (0.7).

An example to Corollary 4 is given by

$$g^{(n)} = \alpha n + \beta, \quad \alpha > 0, \beta > 0.$$

It follows that the continued fraction

$$(3.5) \quad K_{n=0}^{\infty} \frac{(\alpha n + \beta)(\alpha n + \alpha + \beta + 1)}{1}$$

converges to β , regardless of α . This is likely to be known, but is hard to find in the literature. It does not seem to be contained in for instance [11, Chapter VI], where one might expect to find it.

(c)

$$g^{(n)} = -\frac{1}{2} - \frac{L}{4(1+n)}, \quad 0 \leq L \leq 2.$$

We shall find out for which L this particular sequence $g^{(n)}$ is a sequence of right tails for a convergent continued fraction (0.1). Since

$$(3.6) \quad a_n = g^{(n-1)}(1 + g^{(n)}) = -\frac{1}{4} - \frac{(2-L)L}{16n(n+1)}, \quad n \geq 1,$$

we know by [12] (see also [14]) that the continued fraction $K(a_n/1)$ converges for all $L \in [0, 2]$. From [14] we even know the right tails

$$f^{(n)} = -\frac{1}{2} - \frac{1 - \sqrt{1 - (2-L)L}}{4(n+1)} = -\frac{1}{2} - \frac{1 - |1-L|}{4(n+1)},$$

and hence

$$(3.7) \quad f^{(n)} = \begin{cases} -\frac{1}{2} - \frac{L}{4(n+1)} & \text{for } 0 \leq L \leq 1, \\ -\frac{1}{2} - \frac{2-L}{4(n+1)} & \text{for } 1 < L \leq 2. \end{cases}$$

Thus $\{g^{(n)}\}$ is a sequence of right tails if $0 \leq L \leq 1$ and of wrong tails if $1 < L \leq 2$. We shall see that this is what we get by using Theorem 1. We have in this case for $L \leq 1$

$$\kappa_n = \frac{1}{2} - 4(n+1)^{-L/\frac{1}{2}} + 4(n+1)^{-L} \geq \frac{1}{2} - 4(n+1)^{-1/\frac{1}{2}} + 4(n+1)^{-1} = \frac{2n+1}{2n+3},$$

and

$$(3.8) \quad \kappa_1 \kappa_2 \cdots \kappa_n \geq \frac{3}{2n+3}.$$

Hence the κ -series converges to ∞ , and the tails are right. If we replace L by $2-L$, we find for $L > 1$ the inequality (3.8). Hence $-\frac{1}{2} - (2-L)/4(n+1)$ is right for $1 < L \leq 2$ and thus $-\frac{1}{2} - L/4(n+1)$ is wrong.

4. Final remarks. The tails are closely related to the parameters of Wall's chain sequences [15, p. 79], i.e. sequences $(1 - g_0)g_1, (1 - g_1)g_2, (1 - g_2)g_3, \dots$, where $0 \leq g_p \leq 1$. The connection is

$$g_p - 1 = g^{(p)},$$

and

$$(1 - g_p)g_{p+1} = -g^{(p)}(1 + g^{(p+1)}) = -a_{p+1}.$$

In discussing the relations between chain sequences and tails we are thus restricted to tails $g^{(n)} \in [-1, 0]$ and thus $a_n \in [-\frac{1}{4}, 0]$. This, and also relations to results by Chihara [1–3] on chain sequences will be taken up in a later paper.

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