A STRONGER METRIC FOR CLOSED OPERATORS IN HILBERT SPACE¹

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ABSTRACT. There is available, for the closed and densely-defined operators in a Hilbert space, a metric which is stronger than the gap metric while sharing many of its properties. On the bounded operators it is equivalent to the metric generated by the usual operator-norm.

Let H denote an infinite-dimensional Hilbert space, $\mathcal{C}\mathcal{D}(H)$ the family of all closed and densely-defined linear operators in H , and $\mathfrak{B}(H)$ the family of all bounded members of $\mathcal{C}\mathcal{D}(H)$. Let Π denote the function which assigns to each T in $\mathcal{C}\mathcal{D}(H)$ the orthogonal projection from $H \times H$ onto ("the graph of") T . The *gap metric* (cf. [1, p. 197 ff.]) for $\mathcal{C}\mathcal{D}(H)$ may be regarded as arising from Π via the equations

$$d(S, T) = \|\Pi(S) - \Pi(T)\| \quad \text{for all } S \text{ and } T \text{ in } \mathcal{C}\mathcal{D}(H),$$

where $\|\cdot\|$ denotes the operator-norm in $H \times H$. There is available, however, another metric for $\mathcal{C}\mathcal{D}(H)$ which is *stronger* than d while sharing many of its properties. To obtain such a metric, we begin with the function α defined by

$$\alpha(T) = T(1 + T^*T)^{-1/2} \quad \text{for all } T \text{ in } \mathcal{C}\mathcal{D}(H).$$

It is shown in [2, Theorem 2, p. 533], with α identified as Γ^{-1} , that α maps $\mathcal{C}\mathcal{D}(H)$ one-to-one onto the *pure contractions* (which is the family $\mathcal{V}(H)$ of all A in $\mathfrak{B}(H)$ such that $\|Ax\| < \|x\|$ for all nonzero x in H ; here, and in the sequel, $\|\cdot\|$ denotes the norm in H or the operator-norm, according to the context).

Our alternative metric is now definable in terms of α , in a manner exactly analogous to the definition of d in terms of Π , viz.

$$\delta(S, T) = \|\alpha(S) - \alpha(T)\| \quad \text{for all } S \text{ and } T \text{ in } \mathcal{C}\mathcal{D}(H).$$

Clearly $\mathcal{C}\mathcal{D}(H)$ under δ is isometric with the subset $\mathcal{V}(H)$ of the unit ball in $\mathfrak{B}(H)$ under the operator-norm, so that $\delta(S, T) \leq 2$ for all S and T . The purpose of this report is to indicate why δ is stronger than d and to prove some other interesting properties of δ as well. We begin with a useful lemma.

LEMMA 1. *If $\{A_n\}$ is an infinite sequence of bounded nonnegative operators on H , B is in $\mathfrak{B}(H)$, and $\|A_n - B\| \rightarrow 0$ as $n \rightarrow \infty$, then B is nonnegative and $\|A_n^{1/2} - B^{1/2}\| \rightarrow 0$ as $n \rightarrow \infty$.*

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Since this is a special case of the well-known fact that for *any* continuous complex function f on the real line, $\|A_n - B\| \rightarrow 0$ implies $\|f(A_n) - f(B)\| \rightarrow 0$, together with the observation that a limit of nonnegative operators is nonnegative, we omit the proof. (Note that the lemma may be paraphrased as “the square-root function is norm-continuous”.)

THEOREM 1. *The metric δ is stronger than the gap metric d .*

PROOF. For each T in $\mathcal{C}\mathfrak{D}(H)$, let $\beta(T)$ denote the bounded positive operator $(1 + T^*T)^{-1/2}$ (so that $\alpha(T) = T\beta(T)$). As is pointed out in [2, Remark, p. 532], there is an operator-matrix representation for the projection-function Π in terms of α and β , viz.

$$\Pi(T) = \begin{pmatrix} \beta(T)^2 & \beta(T)\alpha(T)^* \\ \alpha(T)\beta(T) & \alpha(T)\alpha(T)^* \end{pmatrix}.$$

Thus for T in $\mathcal{C}\mathfrak{D}(H)$ and an infinite $\mathcal{C}\mathfrak{D}(H)$ -valued sequence $\{S_n\}$, we conclude that $d(S_n, T) \rightarrow 0$ if and only if each of the following conditions holds:

- (i) $\|\beta(S_n)^2 - \beta(T)^2\| \rightarrow 0$;
- (ii) $\|\beta(S_n)\alpha(S_n)^* - \beta(T)\alpha(T)^*\| \rightarrow 0$;
- (iii) $\|\alpha(S_n)\beta(S_n) - \alpha(T)\beta(T)\| \rightarrow 0$; and
- (iv) $\|\alpha(S_n)\alpha(S_n)^* - \alpha(T)\alpha(T)^*\| \rightarrow 0$.

Now suppose, for such $\{S_n\}$ and T , $\delta(S_n, T) \rightarrow 0$, i.e., $\|\alpha(S_n) - \alpha(T)\| \rightarrow 0$. Then, once again by the appropriate norm-continuities, (iv) is true. By [2, Theorem 2 and Proof, p. 533], we can express β in terms of α : $\beta = (1 - \alpha^*\alpha)^{1/2}$, so (i) follows from the “dual” of (iv), that is,

$$\begin{aligned} \|\beta(S_n)^2 - \beta(T)^2\| &= \|[1 - \alpha(S_n)^*\alpha(S_n)] - [1 - \alpha(T)^*\alpha(T)]\| \\ &= \|\alpha(S_n)^*\alpha(S_n) - \alpha(T)^*\alpha(T)\| \rightarrow 0. \end{aligned}$$

We now apply Lemma 1 to obtain $\|\beta(S_n) - \beta(T)\| \rightarrow 0$, from which (ii) and (iii) follow immediately.

At this point the proof is complete if the phrase “is stronger than” is construed so as not to exclude the possibility that δ and d generate the same topology. To see that δ and d are, in fact, not equivalent, we construct an example of a sequence which is d - but not δ -convergent.

EXAMPLE. Assume H is separable and let $\{\phi_n\}$ denote an orthonormal basis therein. For each positive integer n , let C_n denote the member of $\mathfrak{B}(H)$ such that

$$C_n\phi_p = \begin{cases} p\phi_p & \text{if } p \neq n, \\ -p\phi_p & \text{if } p = n. \end{cases}$$

Also, let C denote the member of $\mathcal{C}\mathfrak{D}(H)$ such that $C\phi_p = p\phi_p$ for all p (note that C^{-1} is in $\mathfrak{B}(H)$). Now let $A = \alpha(C)$, $B = \beta(C)$, and for each n , $A_n = \alpha(C_n)$ and $B_n = \beta(C_n)$. Clearly $A_n^2 = A^2$ and $B_n = B$ for all n , so conditions (i)–(iv) in the proof of the theorem hold. Thus $d(C_n, C) \rightarrow 0$. On the other hand, as $n \rightarrow \infty$,

$$\delta(C_n, C) = \|A_n - A\| = 2n(1 + n^2)^{-1/2} \rightarrow 2.$$

A useful property of the metric d is that in $\mathfrak{B}(H)$ it is equivalent to the metric generated by the operator-norm. The metric δ also has this property.

THEOREM 2. *In $\mathfrak{B}(H)$ the metric δ is equivalent to the operator-norm.*

PROOF. A member T of $\mathcal{C}\mathfrak{D}(H)$ is bounded if and only if its domain is H ; in this case $\beta(T)$ is invertible (positive with range H). Now if $\{S_n\}$ is a $\mathfrak{B}(H)$ -valued sequence and $\|S_n - T\| \rightarrow 0$, then $\|\beta(S_n)^{-2} - \beta(T)^{-2}\| = \|T^*T - S_n^*S_n\| \rightarrow 0$ and, thus, by Lemma 1, $\|\beta(S_n)^{-1} - \beta(T)^{-1}\| \rightarrow 0$. Therefore

$$\delta(S_n, T) = \|\alpha(S_n) - \alpha(T)\| = \|S_n\beta(S_n)^{-1} - T\beta(T)^{-1}\| \rightarrow 0.$$

Conversely, if $\|\alpha(S_n) - \alpha(T)\| \rightarrow 0$, we use the relation $\beta = (1 - \alpha^*\alpha)^{1/2}$, coupled with the appropriate continuity of arithmetic (including Lemma 1), to obtain $\|\beta(S_n) - \beta(T)\| \rightarrow 0$ and, hence (recall $\alpha(T) = T\beta(T)^{-1}$),

$$\|S_n - T\| = \|S_n\beta(S_n)^{-1}\beta(S_n) - T\beta(T)^{-1}\beta(T)\| = \|\alpha(S_n)\beta(S_n) - \alpha(T)\beta(T)\| \rightarrow 0.$$

THEOREM 3. *With respect to δ , $\mathfrak{B}(H)$ is open and dense in $\mathcal{C}\mathfrak{D}(H)$.*

PROOF. That $\mathfrak{B}(H)$ is δ -open follows immediately from the recollection that a member T of $\mathcal{C}\mathfrak{D}(H)$ is bounded if and only if $\|\alpha(T)\| < 1$. Now suppose T is in $\mathcal{C}\mathfrak{D}(H)$. Then $\alpha(T)$ is in the closed unit ball of $\mathfrak{B}(H)$, relative to the operator-norm. Hence $\{n(n+1)^{-1}\alpha(T)\}$ is a sequence $\{A_n\}$ of operators such that for each n , $\|A_n\| < 1$ (so A_n is a pure contraction) and $\|A_n - \alpha(T)\| \rightarrow 0$. For each n , let $S_n = \alpha^{-1}(A_n)$; by [2, Theorem 2] each S_n is in $\mathfrak{B}(H)$ and, clearly, $\|\alpha(S_n) - \alpha(T)\| \rightarrow 0$. Hence T is the δ -limit of the $\mathfrak{B}(H)$ -sequence $\{S_n\}$ and the proof is complete.

THEOREM 4. *Suppose each of S and T is in $\mathcal{C}\mathfrak{D}(H)$, and $V|S|$ and $W|T|$ are their respective polar decompositions. Then each of these holds:*

- (1) $\delta(S^*, T^*) = \delta(S, T)$.
- (2) $\delta(|S|, |T|) = \|\alpha(|S|) - \alpha(|T|)\|$.
- (3) $\delta(V, W) = \sqrt{1/2} \|V - W\|$.
- (4) $\delta(S, T) \leq \sqrt{2} \delta(V, W) + \delta(|S|, |T|)$.

PROOF. (1) follows at once from the fact that for all C in $\mathcal{C}\mathfrak{D}(H)$, $\alpha(C^*) = \alpha(C)^*$. For a proof see [3, Lemma 1]. (2) is an immediate corollary of [3, Lemma 3]. To see that (3) is true, note that by the definition of polar decomposition, V and W are partial isometries. Hence by [3, Theorem 3], $\alpha(V) = \sqrt{1/2} V$ and $\alpha(W) = \sqrt{1/2} W$, from which the statement is obvious. Finally, we obtain (4) from (3) and [3, Lemma 3] as follows:

$$\begin{aligned} \delta(S, T) &= \|\alpha(V|S|) - \alpha(W|T|)\| = \|V\alpha(|S|) - W\alpha(|T|)\| \\ &\leq \|V - W\| \cdot \|\alpha(|S|)\| + \|W\| \cdot \|\alpha(|S|) - \alpha(|T|)\| \\ &\leq \sqrt{1/2} \delta(V, W) + \delta(|S|, |T|). \end{aligned}$$

We are now in a position to obtain a connection between δ -convergence and the polar decomposition, namely,

THEOREM 5. *Suppose $\{S_n\}$ is a $\mathcal{C}\mathfrak{D}(H)$ -valued sequence, T is in $\mathcal{C}\mathfrak{D}(H)$, $W|T|$ is the polar decomposition of T , and for each n , $V_n|S_n|$ is the polar decomposition of S_n . Then*

- (1) *if $\delta(S_n, T) \rightarrow 0$ as $n \rightarrow \infty$, then $\delta(|S_n|, |T|) \rightarrow 0$ as well; and*
- (2) *if $\delta(V_n, W) \rightarrow 0$ (equivalently, $\|V_n - W\| \rightarrow 0$) and $\delta(|S_n|, |T|) \rightarrow 0$ as $n \rightarrow \infty$, then $\delta(S_n, T) \rightarrow 0$ as well.*

PROOF. Suppose $\delta(S_n, T) \rightarrow 0$. Then $\|\alpha(S_n)^*\alpha(S_n) - \alpha(T)^*\alpha(T)\| \rightarrow 0$ and thus, by Lemma 1, $\|\alpha(S_n) - \alpha(T)\| \rightarrow 0$. We now get (1) from (2) of Theorem 4. The present statement (2) is an immediate corollary of (4) of Theorem 4, and the proof is complete.

One of the drawbacks of the gap metric is that it is not naturally related to strong convergence (i.e., pointwise) in $\mathcal{C}\mathfrak{D}(H)$. Although the corresponding situation for δ is not much better, there is the following result which, at least for the case in which all operators involved are normal and mutually commuting, seems helpful.

THEOREM 6. *Suppose C is in $\mathcal{C}\mathfrak{D}(H)$ and $\{C_n\}$ is an infinite $\mathcal{C}\mathfrak{D}(H)$ -valued sequence such that for each n , $\text{dom } C \subseteq \text{dom } C_n$. Then for each n the operator $T_n = 1 - \beta(C_n)^{-1}\beta(C)$ is in $\mathfrak{B}(H)$. Moreover, if $\delta(C_n, C) \rightarrow 0$ and, for each x in H , $\|T_n x\| \rightarrow 0$ as $n \rightarrow \infty$, then for all y in $\text{dom } C$, $\|C_n y - C y\| \rightarrow 0$ as $n \rightarrow \infty$.*

PROOF. In [2, Theorem 2 and Proof, p. 533] it is shown that for each S in $\mathcal{C}\mathfrak{D}(H)$, $\beta(S)$ is one-to-one and $\text{ran } \beta(S) = \text{dom } S$. Hence, the condition $\text{dom } C \subseteq \text{dom } C_n$ is equivalent to $\text{ran } \beta(C) \subseteq \text{dom } \beta(C_n)^{-1}$; the latter implies $\text{dom } \beta(C_n)^{-1}\beta(C) = H$ and, thus, T_n belongs to $\mathfrak{B}(H)$. (This last assertion follows from the closed graph theorem and the fact that T_n is a closed operator.) Let y denote a point of $\text{dom } C$. Since $\text{ran } \beta(C) = \text{dom } C$, $\beta(C)^{-1}y$ is a point x of H such that $y = \beta(C)x$. Now for all n we have (recall that $\alpha(S) = S\beta(S)$)

$$\begin{aligned}
 \|(C - C_n)y\| &= \|(C - C_n)\beta(C)x\| = \|[\alpha(C) - C_n\beta(C)]x\| \\
 &\leq \|[\alpha(C) - \alpha(C_n)]x\| + \|[\alpha(C_n) - C_n\beta(C)]x\| \\
 &\leq \delta(C_n, C)\|x\| + \|[\alpha(C_n) - C_n\beta(C_n)\beta(C_n)^{-1}\beta(C)]x\| \\
 &\leq \delta(C_n, C)\|x\| + \|\alpha(C_n)\| \cdot \|[1 - \beta(C_n)^{-1}\beta(C)]x\| \\
 &\leq \delta(C_n, C)\|x\| + \|T_n x\|.
 \end{aligned}$$

Thus, if C_n δ -converges to C and T_n converges strongly to 0, then C_n converges strongly to C on its domain, as was to be proved.

This result suggests that we conclude with a question: is there a metric for the closed and densely-defined operators which is stronger than the gap metric and weaker than strong sequential convergence (in the sense indicated in the theorem)?

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